Computation and Modeling Assignment 23

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Problem 23-2

1. Part 1: Consider the general exponential distribution defined by

$$p_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   (a) Using integration, show that this is a valid distribution, i.e. all the probability integrates to 1.

   Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} p_\lambda(x)\mathrm{d}x &= \int_0^{\infty} \lambda e^{-\lambda x}\mathrm{d}x \\ &= -e^{-\lambda x}\big|_0^{\infty} \\ &= 0 - (-1) \\ &= 1 \end{aligned}$$

   (b) Given that $X \sim p_\lambda$, compute $P(0 < X < 1)$.

   Solution:

$$\begin{aligned} \int_0^1 p_\lambda(x)\mathrm{d}x &= \int_0^1 \lambda e^{-\lambda x}\mathrm{d}x \\ &= -e^{-\lambda x}\big|_0^1 \\ &= -e^{-\lambda} + 1 \end{aligned}$$

   (c) Given that $X \sim p_\lambda$, compute $E[X]$.

   Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} x p_\lambda(x)\mathrm{d}x &= \int_0^{\infty} \lambda x e^{-\lambda x}\mathrm{d}x \\ &= -e^{-\lambda x}(x + \frac{1}{\lambda})\big|_0^{\infty} \\ &= 0 - (-\frac{1}{\lambda}) \\ &= \frac{1}{\lambda} \end{aligned}$$

   (d) Given that $X \sim p_\lambda$, compute $\mathrm{Var}[X]$.

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty} p_\lambda(x)(x - E[X])^2 \mathrm{d}x &= \int_0^\infty \lambda e^{-\lambda x}(x - \frac{1}{\lambda})^2 \mathrm{d}x \\ &= -e^{-\lambda x}((x - \frac{1}{\lambda})^2 + \frac{2}{\lambda}(x - \frac{1}{\lambda}) + \frac{2}{\lambda^2})\Big|_0^\infty \\ &= -e^{-\lambda x}(x^2 + \frac{1}{\lambda^2})\Big|_0^\infty \\ &= 0 - (-\frac{1}{\lambda^2}) \\ &= \frac{1}{\lambda^2}\end{aligned}$$

2. Part 2: Consider the general uniform distribution on the interval $[a, b]$. It takes the following form for some constant k:

$$p(x) = \begin{cases} k & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$

(a) Find the value of k such that $p(x)$ is a valid probability distribution. Your answer should be in terms of a and b.

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty} p(x)\mathrm{d}x = \int_a^b k\mathrm{d}x &= 1 \\ kx\Big|_a^b &= 1 \\ bk - ak &= 1 \\ (b-a)k &= 1 \\ k &= \frac{1}{b-a}\end{aligned}$$

(b) Given that $X \sim p$, compute the cumulative distribution $P(X \leq x)$. Your answer should be a piecewise function.

Solution:

$$P(X \leq x) = \int_{-\infty}^{x} p(t)\mathrm{d}t$$

For $x < a$:

$$P(X \leq x) = \int_{-\infty}^{x} p(t)\mathrm{d}t = \int_{-\infty}^{x} 0\mathrm{d}t = 0$$

For $a < x < b$:

$$\begin{aligned}P(X \leq x) &= \int_a^x p(t)\mathrm{d}t = \int_a^x \frac{1}{b-a}\mathrm{d}t \\ &= \frac{t}{b-a}\Big|_a^x \\ &= \frac{x}{b-a} - \frac{a}{b-a} \\ &= \frac{x-a}{b-a}\end{aligned}$$

For $x > b$:

$$P(X \leq x) = \int_b^x p(t)\mathrm{d}t = \int_b^x 0\mathrm{d}t = 0$$

$$P(X \leq x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x \leq b \\ 0 & x > b \end{cases}$$

(c) Given that $X \sim p$, compute $E[X]$.

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} xp(x)\mathrm{d}x &= \int_a^b \frac{x}{b-a}\mathrm{d}x \\ &= \frac{x^2}{2(b-a)}\Big|_a^b \\ &= \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} \\ &= \frac{b^2-a^2}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

(d) Given that $X \sim p$, compute $\mathrm{Var}[X]$.

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty} p(x)(x-E[X])^2\mathrm{d}x &= \int_a^b k(x-E[X])^2\mathrm{d}x \\ &= \int_a^b \frac{(x-\frac{b+a}{2})^2}{b-a}\mathrm{d}x \\ &= \frac{(x-\frac{b+a}{2})^3}{3(b-a)}\Big|_a^b \\ &= \frac{(\frac{b-a}{2})^3}{3(b-a)} - \frac{(\frac{a-b}{2})^3}{3(b-a)} \\ &= \frac{(b-a)^2}{12} \end{aligned}$$