Computation and Modeling Assignment 27

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Problem 27-3

Suppose you toss a coin 10 times and get the result HHHHT HHHHH. From this result, you estimate that the coin is biased and generally lands on heads 90% of the time. But how sure can you be? Let's quantify it.

a) Compute the likelihood $P(\text{HHHHTHHHHH}|k)$ where $P(\text{H}) = k$. Remember that the likelihood is just the probability of getting the result HHHHT HHHHH under the assumption that $P(\text{H}) = k$.

Solution:

$$P(\text{HHHHTHHHHH}|k) = k^9(1-k) = k^9 - k^{10}$$

b) Create a probability distribution $P(k|\text{HHHHTHHHHH})$ that is proportional to the likelihood $P(\text{HHHHTHHHHH}|k)$. In other words, find the function $P(k)$ such that

$$P(k) = c \cdot P(\text{HHHHTHHHHH}|k)$$

for some constant c, and $\int_0^1 P(k|\text{HHHHTHHHHH}) = 1$

Solution:

$$\begin{aligned}\int_0^1 P(k|\text{HHHHTHHHHH}) = \int_0^1 c(k^9 - k^{10}) = 1 \\ c\left(\frac{k^{10}}{10} - \frac{k^{11}}{11}\right)\Big|_0^1 = 1 \\ \frac{c}{110} = 1 \\ c = 110\end{aligned}$$

The distribution is then:

$$P(k) = 110(k^9 - k^{10})$$

c) Using the prior distribution $P(k) \sim \mathcal{U}[0,1]$, what was the prior probability that the coin was biased towards heads? In other words, what was $P(k > 0.5)$?

Solution:

$$P(k) = \mathcal{U}[0,1] = \begin{cases} 1, & k \in [0,1] \\ 0, & k \notin [0,1] \end{cases}$$

$$\begin{aligned}\int_{0.5}^1 \mathcal{U}[0,1] &= \int_{0.5}^1 1 \\ &= k\Big|_{0.5}^1 \\ &= 0.5\end{aligned}$$

d) Using the posterior distribution $P(k|\text{HHHHTHHHHH})$, what was the posterior probability that the coin was biased towards heads? In other words, what is $P(k > 0.5|\text{HHHHTHHHHH})$?

Solution:

$$\begin{aligned}
\int_{0.5}^{1} P(k|\text{HHHHTHHHHH}) &= \int_{0.5}^{1} 110(k^9 - k^{10}) \\
&= 110\left(\frac{k^{10}}{10} - \frac{k^{11}}{11}\right)\Big|_{0.5}^{1} \\
&= 1 - 0.005859375 \\
&= 0.994140625
\end{aligned}$$

e) Compare your answers in parts (c) and (d). Did the probability that the coin was biased towards heads increase or decrease, after observing the sequence of flips? Why does this make intuitive sense?

Solution: It increased. This makes sense because there were significantly more heads than there were tails in the sample, suggesting that the coin is biased towards heads.

f) Using the posterior distribution, what is the most probable value of k? In other words, what is value of k at which $P(k|\text{HHHHTHHHHH})$ reaches a maximum? Show your work using the first or second derivative test.

Solution: The critical points of $P'(k|\text{HHHHTHHHHH})$ are:

$$\begin{aligned}
P'(k|\text{HHHHTHHHHH}) = 110(9k^8 - 10k^9) &= 0 \\
k^8(9 - 10k) &= 0 \\
k &= 0, 0.9
\end{aligned}$$

The second derivative of $P(k|\text{HHHHTHHHHH})$ is:

$$P''(k|\text{HHHHTHHHHH}) = 110(72k^7 - 90k^8)$$

$k = 0$:

$$P''(k = 0|\text{HHHHTHHHHH}) = 0$$

$k = 0.9$

$$P''(k = 0.9|\text{HHHHTHHHHH}) = -473.513931$$

0.9 is the maximum.

g) Why does your answer to (f) make sense? What's the intuition here?

Solution: It makes sense because 9 out of the 10 flips were heads, so we would expect k to equal 9/10.

h) What is the probability that the bias k lies within 0.05 of your answer to part (g)? In other words, what is the probability that $0.85 < k < 0.95$?

Solution:

$$\begin{aligned}
\int_{0.85}^{0.95} P(k|\text{HHHHTHHHHH}) &= \int_{0.85}^{0.95} 110(k^9 - k^{10}) \\
&= 110\left(\frac{k^{10}}{10} - \frac{k^{11}}{11}\right)\Big|_{0.85}^{0.95} \\
&= 0.898105 - 0.492186 \\
&= 0.405919
\end{aligned}$$

i) Fill in the blank: you can be 99% sure that $P(\mathrm{H})$ is at least ___.

Solution:

$$\begin{aligned}
\int_{k_{min}}^{1} P(k|\mathrm{HHHHTHHHHH}) = \int_{k_{min}}^{1} 110(k^9 - k^{10}) &= 99\% \\
110\left(\frac{k^{10}}{10} - \frac{k^{11}}{11}\right)\Big|_{k_{min}}^{1} &= 0.99 \\
1 - 110\left(\frac{k_{min}^{10}}{10} - \frac{k_{min}^{11}}{11}\right) &= 0.99 \\
110\left(\frac{k_{min}^{10}}{10} - \frac{k_{min}^{11}}{11}\right) &= 0.01 \\
k_{min} &\approx 0.5302
\end{aligned}$$