Computation and Modeling Assignment 35

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Problem 35-1

Your friend is randomly stating positive integers that are less than some upper bound (which your friend knows, but you don't know). The numbers your friend states are as follows:

$$1, 17, 8, 25, 3$$

You assume that the numbers come from a discrete uniform distribution $U\{1, 2, ..., k\}$ defined as follows:

$$p_k(x) = \begin{cases} \frac{1}{k} & x \in \{1, 2, ..., k\} \\ 0 & x \notin \{1, 2, ..., k\} \end{cases}$$

a) Compute the likelihood $P(\{1, 17, 8, 25, 3\}|k)$. Remember that the likelihood is just the probability of getting the result $\{1, 17, 8, 25, 3\}$ under the assumption that the data was sampled from the distribution $p_k(x)$. Your answer should be a piecewise function expressed in terms of k:

Solution:

$$P(\{1, 17, 8, 25, 3\}|k) = \begin{cases} \frac{1}{k^5} & k \geq 25 \\ 0 & \text{otherwise} \end{cases}$$

b) Compute the posterior distribution by normalizing the likelihood. That is to say, find the constant c such that

$$\sum_{k=1}^{\infty} c \cdot P(\{1, 17, 8, 25, 3\}|k) = 1$$

Solution:

$$\sum_{k=1}^{\infty} c \cdot P(\{1, 17, 8, 25, 3\}|k) = 1$$

$$c \sum_{k=25}^{\infty} \frac{1}{k^5} = 1$$

$$6.9290476e - 7c = 1$$

$$c = 1443199.784$$

$$P(k|\{1, 17, 8, 25, 3\}) = c \cdot P(\{1, 17, 8, 25, 3\}|k) = \frac{1443199.784}{k^5}$$

c) What is the most probable value of k? You can tell this just by looking at the distribution $P(k|\{1, 17, 8, 25, 3\})$, but make sure to justify your answer with an explanation.

Solution: 25 is the most probable, as k increases the probability decreases.

d) The largest number in the dataset is 25. What is the probability that 25 is actually the upper bound chosen by your friend?

Solution:

$$P(k = 25|\{1, 17, 8, 25, 3\}) = \frac{1443199.784}{25^5} = 0.14778$$

e) What is the probability that the upper bound is less than or equal to 30?

Solution:

$$P(k = 25|\{1, 17, 8, 25, 3\}) + P(k = 26|\{1, 17, 8, 25, 3\}) + P(k = 27|\{1, 17, 8, 25, 3\})+$$
$$P(k = 28|\{1, 17, 8, 25, 3\}) + P(k = 29|\{1, 17, 8, 25, 3\}) + P(k = 30|\{1, 17, 8, 25, 3\}) = 0.58344$$

f) Fill in the blank: you can be 95% sure that the upper bound is less than 52.

Problem 35-2

The joint uniform distribution $\mathcal{U}([a, b]\times[c, d])$ is a distribution such that all points (x, y) have equal probability in the region $[a, b] \times [c, d]$ and zero probability elsewhere. So, it takes the form

$$p(x, y) = \begin{cases} k & (x, y) \in [a, b] \times [c, d] \\ 0 & (x, y) \notin [a, b] \times [c, d] \end{cases}$$

a) Find the value of k such that $p(x, y)$ is a valid probability distribution. Your answer should be in terms of a, b, c, d.

Solution:

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y)\ \mathrm{d}x\mathrm{d}y = \int_c^d \int_a^b k\ \mathrm{d}x\mathrm{d}y &= 1 \\ \int_c^d k(b-a)\ \mathrm{d}y &= 1 \\ k(b-a)(d-c) &= 1 \\ k &= \frac{1}{(b-a)(d-c)} \end{aligned}$$

b) Given that $(X, Y) \sim p$, compute $E[X]$ and $E[Y]$. You should get $E[X] = \frac{a+b}{2}$ and $E[Y] = \frac{c+d}{2}$

Solution:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xp(x, y)\ \mathrm{d}x\mathrm{d}y \\ &= \int_c^d \int_a^b \frac{x}{(b-a)(d-c)}\ \mathrm{d}x\mathrm{d}y \\ &= \int_c^d \frac{(b^2-a^2)}{2(b-a)(d-c)}\ \mathrm{d}y \\ &= \frac{(b+a)(b-a)(d-c)}{2(b-a)(d-c)} \\ &= \frac{(b+a)}{2} \end{aligned}$$

$$
\begin{aligned}
E[Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yp(x,y)\ \mathrm{d}y\mathrm{d}x \\
&= \int_a^b \int_c^d \frac{y}{2(b-a)(d-c)}\ \mathrm{d}y\mathrm{d}x \\
&= \int_a^b \frac{(d^2-c^2)}{2(b-a)(d-c)}\ \mathrm{d}x \\
&= \frac{(d+c)(d-c)(b-a)}{2(b-a)(d-c)} \\
&= \frac{(d+c)}{2}
\end{aligned}
$$

c) Geometrically, $[a,b]\times[c,d]$ represents a rectangle bounded by $x=a$, $x=b$, $y=c$, and $y=d$. What is the geometric interpretation of the point $(E[X],E[Y])$ in this rectangle?

Solution: The "middle" point (Where the diagonals of the rectangle intersect).