Computation and Modeling Assignment 19

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Problem 1 Part 1

(a) *Write the probability distribution $p_4(n)$ for getting n heads on 4 coin flips, where the coin is a fair coin (i.e. it lands on heads with probability 0.5 .*
(b) *Let N be the number of heads in 4 coin flips. Then $N \sim p_4$. Intuitively, what is the expected value of N? Explain the reasoning behind your intuition.*
(c) *Compute the expected value of N, using the definition $E[N] = \Sigma n \cdot p(n)$. The answer you get should match your answer from (b).*
(d) *Compute the variance of N, using the definition $Var[N] = E[(N - E[N])^2]$. Your answer should come out to 1.*

Solution

(a) Generally, $p_4(n) = \binom{4}{n} \cdot (\frac{1}{2})^4$. The probability distribution would then be $[\binom{4}{0} \cdot (\frac{1}{2})^4, \binom{4}{1} \cdot (\frac{1}{2})^4, \binom{4}{2} \cdot (\frac{1}{2})^4, \binom{4}{3} \cdot (\frac{1}{2})^4, \binom{4}{4} \cdot (\frac{1}{2})^4]$. This can be simplified to $[1 \cdot (\frac{1}{2})^4, 4 \cdot (\frac{1}{2})^4, 6 \cdot (\frac{1}{2})^4, 4 \cdot (\frac{1}{2})^4, 1 \cdot (\frac{1}{2})^4]$. The final probability distribution would be $[\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16}]$

(b) Intuitively, the expected value of N should be 2. My reasoning is that the probability states that we have a 50% chance to land heads. So in a series of 4 coin flips, about 50%, or 2, of them result in the coin landing on heads.

(c) $E[N] = \Sigma n \cdot p(n) = 0 \cdot \frac{1}{16} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{16}$
$= \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4} = 2$

(d)$\text{Var}[N] = E[(N - E[N])^2]$
$\text{Var}[N] = E[(N-2)^2] = \Sigma(n-2)^2 \cdot p(n)$
$= (0-2)^2 \cdot \frac{1}{16} + (1-2)^2 \cdot \frac{1}{4} + (2-2)^2 \cdot \frac{3}{8} + (3-2)^2 \cdot \frac{1}{4} + (4-2)^2 \cdot \frac{1}{16}$
$= 4 \cdot \frac{1}{16} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{3}{8} + 1 \cdot \frac{1}{4} + 4 \cdot \frac{1}{16}$
$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$
$= 1$

Problem 1 Part 2

(a) *Write the probability distribution $p_{4,k}(n)$ for getting n heads on 4 coin flips, where the coin is a biased coin that lands on heads with probability k.*

If you substitute $k = 0.5$, you should get the same result that you did in part 1(a).

(b) *Let N be the number of heads in 4 coin flips of a biased coin. Then $N \sim p_{4,k}$. Intuitively, what is the expected value of N? Your answer should be in terms of k. Explain the reasoning behind your intuition.*

If you substitute $k = 0.5$, you should get the same result that you did in part 1(b).

(c) *Compute the expected value of N, using the definition $E[N] = \Sigma n \cdot p(n)$.*

The answer you get should match your answer from (b).

Solution

(a) Generally, $p_{n,k}(n) = \binom{4}{n} \cdot k^n \cdot (1-k)^{4-n}$. The probability distribution would then be $[\binom{4}{0} \cdot k^0 \cdot (1-k)^{4-0}, \binom{4}{1} \cdot k^1 \cdot (1-k)^{4-1}, \binom{4}{2} \cdot k^2 \cdot (1-k)^{4-2}, \binom{4}{3} \cdot k^3 \cdot (1-k)^{4-3}, \binom{4}{4} \cdot k^4 \cdot (1-k)^{4-4}]$. This can be simplified to $[1 \cdot k^0 \cdot (1-k)^4, 4 \cdot k^1 \cdot (1-k)^3, 6 \cdot k^2 \cdot (1-k)^2, 4 \cdot k^3 \cdot (1-k)^1, 1 \cdot k^4 \cdot (1-k)^0]$. Simplifying further, we get $[(1-k)^4, 4k^1 \cdot (1-k)^3, 6k^2 \cdot (1-k)^2, 4k^3 \cdot (1-k)^1, k^4 \cdot (1-k)^0]$. Finally, we get $[(1-k)^4, 4k^1 \cdot (1-k)^3, 6k^2 \cdot (1-k)^2, 4k^3 \cdot (1-k)^1, k^4]$

(b) The expected value of N should be $4k$. In the last example, the expected value was 2 since 2 was 50% of 4. This time, we have $k \cdot 100\%$ instead of 50%. Hence, we get 4k.

(c) $E[N] = 0 \cdot (1-k)^4 + 1 \cdot 4k^1 \cdot (1-k)^3 + 2 \cdot 6k^2 \cdot (1-k)^2 + 3 \cdot 4k^3 \cdot (1-k)^1 + 4 \cdot k^4$
$= 4k \cdot (1 - 3k + 3k^2 - k^3) + 12k^2 \cdot (1 - 2k + k^2) + 12k^3 \cdot (1-k) + 4 \cdot k^4$
$= (4k - 12k^2 + 12k^3 - 4k^4) + (12k^2 - 24k^3 + 12k^4) + (12k^3 - 12k^4) + 4k^4$
$= 4k$