Computation Modeling Assignment 21

Cayden Lau

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Problem 21-2

(a) *Find the value of k such that $p(x)$ is a valid probability distribution. (Remember that for a function to be a valid probability distribution, it must integrate to 1.)*
(b) *Given that $X \sim U[3,7]$, compute $E[X]$.*

Check: does your result make intuitive sense? If you pick a bunch of numbers from the interval $[3,7]$, and all of those numbers are equally likely choices, then what would you expect to be the average of the numbers you pick?
(c) *Given that $X \sim U[3,7]$, compute $Var[X]$.*

You should get $\frac{4}{3}$.

Solution

(a) $\int_{-\infty}^{\infty} p(x)\,dx$

$= \int_{-\infty}^{3} p(x)\,dx + \int_{3}^{7} p(x)\,dx + \int_{7}^{\infty} p(x)\,dx$
$= \int_{-\infty}^{3} 0\,dx + \int_{3}^{7} k\,dx + \int_{7}^{\infty} 0\,dx$
$= \int_{3}^{7} k\,dx$
$= k(7) - k(3)$
$= 4k$
Remember, for this to be a VALID probability distribution, it must integrate to 1, so
$4k = 1$
$k = \frac{1}{4}$

(b) $E[X] = \int_{-\infty}^{\infty} x \cdot p(x)\,dx$

$= \int_{-\infty}^{3} x \cdot p(x)\,dx + \int_{3}^{7} x \cdot p(x)\,dx + \int_{7}^{\infty} x \cdot p(x)\,dx$
$= \int_{-\infty}^{3} x \cdot 0\,dx + \int_{3}^{7} x \cdot \frac{1}{4}\,dx + \int_{7}^{\infty} x \cdot 0\,dx$
$= \int_{3}^{7} x \cdot \frac{1}{4}\,dx$

$= \left.\dfrac{x^2}{8}\right|_3^7$
$= \frac{49}{8} - \dfrac{9}{8}$
$= \frac{40}{8}$
$= 5$
This makes sense because 5 is also the median and mean of the set $[3, 4, 5, 6, 7]$.

(c) Var[X] $= E[(X - E[X])^2]$

$= \int\limits_{-\infty}^{\infty} (x-5)^2 \cdot p(x)\,\mathrm{d}x$
$= \int\limits_{-\infty}^{3} (x-5)^2 \cdot p(x)\,\mathrm{d}x + \int\limits_{3}^{7} (x-5)^2 \cdot p(x)\,\mathrm{d}x + \int\limits_{7}^{\infty} (x-5)^2 \cdot p(x)\,\mathrm{d}x$
$= \int\limits_{-\infty}^{3} (x-5)^2 \cdot 0\,\mathrm{d}x + \int\limits_{3}^{7} (x-5)^2 \cdot \frac{1}{4}\,\mathrm{d}x + \int\limits_{7}^{\infty} (x-5)^2 \cdot 0\,\mathrm{d}x$
$= \int\limits_{3}^{7} (x-5)^2 \cdot \frac{1}{4}\,\mathrm{d}x$
$= \left.\dfrac{1}{4} \cdot (\frac{x^3}{3} - 5x^2 + 25x)\right|_3^7$
$= \frac{1}{4} \cdot ((\frac{343}{3} - 245 + 175) - (\frac{27}{3} - 45 + 75)$
$= \frac{1}{4} \cdot (\frac{316}{3} - 200 + 100)$
$= \frac{1}{4} \cdot (\frac{316}{3} - 100)$
$= \frac{1}{4} \cdot (\frac{316}{3} - \frac{300}{3})$
$= \frac{1}{4} \cdot (\frac{16}{3})$
$= \frac{4}{3}$