Computation Modeling Assignment 23

Cayden Lau

February 27, 2021

Problem 1

(a) *Using integration, show that this is a valid distribution, i.e. all the probability integrates to 1.*
(b) *Given that* $X \sim p_\lambda$, *compute* $P(0 < X < 1)$.
(c) *Given that* $X \sim p_\lambda$, *compute* $\mathrm{E}[X]$.
(d) *Given that* $X \sim p_\lambda$, *compute* $\mathrm{Var}[X]$.

Problem 2

(a) *Find the value of* k *such that* $p(x)$ *is a valid probability distribution. Your answer should be in terms of* a *and* b.
(b) *Given that* $X \sim p$, *compute the cumulative distribution* $P(X \leq x)$. *Your answer should be a piecewise function:*
(c) *Given that* $X \sim p$, *compute* $\mathrm{E}[X]$.

$$P(X \leq x) = \begin{cases} \_ & \text{if } x < a \\ \_ & \text{if } a \leq x \leq b \\ \_ & \text{if } b < x \end{cases}$$

(d) *Given that* $X \sim p_\lambda$, *compute* $\mathrm{Var}[X]$.

Problem 1 Solutions

(a)

$$\begin{aligned}
\int_{-\infty}^{\infty} p_\lambda(x)\ \mathrm{d}x &= \int_{-\infty}^{0} p_\lambda(x)\ \mathrm{d}x + \int_{0}^{\infty} p_\lambda(x)\ \mathrm{d}x \\
&= \int_{-\infty}^{0} 0\ \mathrm{d}x + \int_{0}^{\infty} \lambda e^{-\lambda x}\ \mathrm{d}x \\
&= \left(-e^{-\lambda x}\right)\Big|_0^\infty \\
&= -e^{-\lambda\cdot\infty} - (-e^{-\lambda\cdot 0}) \\
&= 1
\end{aligned}$$

(b)

$$\begin{aligned}
P(0 < X < 1) &= \int_0^1 p_\lambda(x)\,\mathrm{d}x \\
&= \int_0^1 \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= \left(-e^{-\lambda x}\right)\Big|_0^1 \\
&= -e^{-\lambda\cdot 1} + e^{-\lambda\cdot 0} \\
&= 1 - e^{-\lambda}
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{E}[X] &= \int_{-\infty}^{\infty} x \cdot p_\lambda(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{0} x \cdot p_\lambda(x)\,\mathrm{d}x + \int_{0}^{\infty} x \cdot p_\lambda(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{0} x \cdot 0\,\mathrm{d}x + \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= -e^{-\lambda x}x - \frac{e^{-\lambda x}}{\lambda}\Bigg|_0^\infty \\
&= -e^{-\lambda\cdot\infty}\cdot\infty - \frac{e^{-\lambda\cdot\infty}}{\lambda} - \left(-e^{-\lambda\cdot 0}\cdot 0 - \frac{e^{-\lambda\cdot 0}}{\lambda}\right) \\
&= \frac{1}{\lambda}
\end{aligned}$$

(d)

$$
\begin{aligned}
\mathrm{Var}[X] &= \int_{-\infty}^{\infty} \left(\left(x - \frac{1}{\lambda} \right)^2 \cdot p_\lambda(x) \right) \mathrm{d}x \\
&= \int_{-\infty}^{0} \left(x - \frac{1}{\lambda} \right)^2 \cdot p_\lambda(x) \,\mathrm{d}x + \int_{0}^{\infty} \left(x - \frac{1}{\lambda} \right)^2 \cdot p_\lambda(x) \,\mathrm{d}x \\
&= \int_{-\infty}^{0} \left(x - \frac{1}{\lambda} \right)^2 \cdot 0 \,\mathrm{d}x + \int_{0}^{\infty} \left(x - \frac{1}{\lambda} \right)^2 \cdot \lambda e^{-\lambda x} \,\mathrm{d}x \\
&= \left(-\frac{e^{-\lambda x}}{\lambda^2} - e^{-\lambda x} x^2 \right) |_0^\infty \\
&= -\frac{e^{-\lambda \cdot \infty}}{\lambda^2} - e^{-\lambda \cdot \infty} \cdot \infty^2 - \left(-\frac{e^{-\lambda \cdot 0}}{\lambda^2} - e^{-\lambda \cdot 0} \cdot 0^2 \right) \\
&= \frac{1}{\lambda^2}
\end{aligned}
$$

Problem 2 Solutions

(a)

$$
\begin{aligned}
\int_{-\infty}^{\infty} p(x) \,\mathrm{d}x &= \int_{-\infty}^{a} p(x) \,\mathrm{d}x + \int_{a}^{b} p(x) \,\mathrm{d}x + \int_{b}^{\infty} p(x) \,\mathrm{d}x \\
1 &= \int_{-\infty}^{a} 0 \,\mathrm{d}x + \int_{a}^{b} k \,\mathrm{d}x + \int_{b}^{\infty} 0 \,\mathrm{d}x \\
1 &= \int_{a}^{b} k \,\mathrm{d}x \\
1 &= kx|_a^b \\
1 &= kb - ka \\
1 &= k(b-a) \\
k &= \left(\frac{1}{b-a} \right)
\end{aligned}
$$

(b)

$$
P(X \leq x) = \begin{cases} 0 & \text{if } x < a \\ \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{if } b < x \end{cases}
$$

(c)

$$
\begin{aligned}
\mathrm{E}[X] &= \int_{-\infty}^{\infty} x \cdot p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{a} x \cdot p(x)\,\mathrm{d}x + \int_{a}^{b} x \cdot p(x)\,\mathrm{d}x + \int_{b}^{\infty} x \cdot p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{a} x \cdot 0\,\mathrm{d}x + \int_{a}^{b} x \cdot \frac{1}{b-a}\,\mathrm{d}x + \int_{b}^{\infty} x \cdot 0\,\mathrm{d}x \\
&= \int_{a}^{b} x \cdot \frac{1}{b-a}\,\mathrm{d}x \\
&= \left. \frac{x^2}{2(b-a)} \right|_a^b \\
&= \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} \\
&= \frac{b^2-a^2}{2(b-a)} \\
&= \frac{b+a}{2}
\end{aligned}
$$

(d)

$$
\begin{aligned}
\mathrm{Var}[X] &= \int_{-\infty}^{\infty} \left(x - \frac{b+a}{2}\right)^2 \cdot p(x)\,\mathrm{d}x \\
&= \int_{a}^{b} \left(x - \frac{b+a}{2}\right)^2 \cdot \frac{1}{b-a}\,\mathrm{d}x \\
&= \left. \frac{4x^3 - 6bx^2 + 6ax^2 + 3x\,(b-a)^2}{12\,(b-a)} \right|_a^b \\
&= \frac{b^2 - 2ba + a^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$