Computation Modeling Assignment 27

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Problem 27-3

(a) *Compute the likelihood* $P(HHHHT\ HHHHH \mid k)$ *where* $P(H) = k$. *Remember that the likelihood is just the probability of getting the result HHHHT HHHHH under the assumption that* $P(H) = k$. *Your answer should be expressed in terms of* k.
(b) *The likelihood* $P(HHHHT\ HHHHH \mid k)$ *can almost be interpreted as a probability distribution for* k. *The only problem is that it doesn't integrate to 1. Create a probability distribution* $P(k \mid HHHHT\ HHHHH)$ *that is proportional to the likelihood* $P(HHHHT\ HHHHH \mid k)$. *In other words, find the function* $P(k)$ *such that*

$$P(k \mid \text{HHHHT HHHHH}) = c \cdot P(k \mid \text{HHHHT HHHHH})$$

For some constant c, and $\int_0^1 P(k \mid \text{HHHHT HHHHH}) = 1$
(c) *Using the prior distribution* $P(k) \sim U[0,1]$, *what was the prior probability that the coin was biased towards heads? In other words, what was* $P(k > 0.5)$*?*
(d) *Using the posterior distribution* $P(k \mid HHHHT\ HHHHH)$, *what was the posterior probability that the coin was biased towards heads? In other words, what is* $P(k > 0.5 \mid HHHHT\ HHHHH)$*?*
(e) *Compare your answers in parts (c) and (d). Did the probability that the coin was biased towards heads increase or decrease, after observing the sequence of flips? Why does this make intuitive sense?*
(f) *Using the posterior distribution, what is the most probable value of k? In other words, what is value of k at which* $P(k \mid HHHHT\ HHHHH)$ *reaches a maximum? Show your work using the first or second derivative test.*
(g) *Why does your answer to (f) make sense? What's the intuition here?*
(h) *What is the probability that the bias k lies within* 0.05 *of your answer to part (g)? In other words, what is the probability that* $0.85 < k < 0.95$*?*
(i) *Fill in the blank: you can be 99% sure that* $P(H)$ *is at least* ___.

Solution

(a)

$$\begin{aligned} P(\text{HHHHT HHHHH} \mid k) &= P(H) \cdot P(H) \cdot P(H) \cdot P(H) \cdot P(T) \cdot P(H) \cdot P(H) \cdot P(H) \cdot P(H) \cdot P(H) \\ &= k \cdot k \cdot k \cdot k \cdot (1-k) \cdot k \cdot k \cdot k \cdot k \cdot k \\ &= k^9 \cdot (1-k) \\ &= k^9 - k^{10} \end{aligned}$$

(b)

$$\begin{aligned} P(\text{HHHHT HHHHH} \mid k) &= \int_0^1 k^9 - k^{10} \\ &= \frac{k^{10}}{10} - \frac{k^{11}}{11}\bigg|_0^1 \\ \frac{1}{10} - \frac{1}{11} \\ &= \frac{1}{110} \end{aligned}$$

So then posterior distribution

$$\begin{aligned} P(k \mid \text{HHHHT HHHHH}) &= 110 \cdot P(\text{HHHHT HHHHH} \mid k) \\ &= 110k^9 - 110k^{10} \end{aligned}$$

(c)

$$\begin{aligned} P(k > 0.5) &= \int_0^{0.5} P(k) \\ &= \int_0^{0.5} U[0.1] \\ &= 0.50 \end{aligned}$$

(d)

$$\begin{aligned} P(k > 0.5 \mid \text{HHHHT HHHHH}) &= \int_0^{0.5} P(k) \\ &= \int_{0.5}^1 110k^9 - 110k^{10} \\ &= 11k^{10} - 10k^{11}\Big|_{0.5}^1 \\ &= 1 - 0.005859375 \\ &= 0.9941 \end{aligned}$$

(e) The probability that the coin was biased towards heads increased. Intuitively, it makes sense because there are more heads than tails in the flip sequence. Intuitively, the number of tails should be more or less the number of heads, but this isn't the case. However, since we have quite a few more heads than tails, the probability increasing makes intuitive sense.

(f)

$$\begin{aligned} P'(k) &= 990k^8 - 1100k^9 \\ 0 &= 990k^8 - 1100k^9 \\ \text{So then our extramum is at } k &= 0.9 \\ P''(k) &= 7920k^7 - 9900k^8 \\ P''(0.9) &= -473.513931 \end{aligned}$$

Which means that $k = 0.9$ is where we have our maximum, making 0.9 the most probably value for k

(g) The solution for (f) makes intuitive sense because 9 out of 10 of our flips were heads. Intuition leads us to think that our chances for getting heads is then 90%

(h)

$$\begin{aligned} P(0.85 < k < 0.95 \,|\, \text{HHHHT HHHHH}) - & \int_{0.85}^{0.95} P(k) \\ &= \int_{0.85}^{0.95} 110k^9 - 110k^{10} \\ &= 11k^{10} - 10k^{11}\Big|_{0.85}^{0.95} \\ &= 0.4059 \end{aligned}$$

(i)

$$\begin{aligned} 0.99 &= \int_n^1 110k^9 - 110k^{10} \\ 0.99 &= 11k^{10} - 10k^{11}\Big|_n^1 \\ 0.99 &= 1 - \left(11n^{10} - 10n^{11}\right) \\ n &= -0.7463, 0.5302, 1.012 \end{aligned}$$

However, since we are working within [0, 1], we can conclude that $n = 0.9859$. So then $P(H)$ is at least 0.5302.