Computation Modeling Assignment 33

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Problem 33-1

Solution

Step 1:

$$\begin{aligned} y &= \frac{1}{1+e^{ax+b}} \\ 1+e^{ax+b} &= \frac{1}{y} \\ e^{ax+b} &= \frac{1}{y}-1 \\ ax+b &= \ln\left|\frac{1}{y}-1\right| \end{aligned}$$

Step 2:

$$\begin{aligned} a+b &= \ln\left|\frac{1}{0.2}-1\right| \\ 2a+b &= \ln\left|\frac{1}{0.25}-1\right| \\ 3a+b &= \ln\left|\frac{1}{0.5}-1\right| \\ \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} \ln\left|\frac{1}{0.2}-1\right| \\ \ln\left|\frac{1}{0.25}-1\right| \\ \ln\left|\frac{1}{0.5}-1\right| \end{bmatrix} \end{aligned}$$

Step 3:

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \ln\left|\frac{1}{0.2} - 1\right| \\ \ln\left|\frac{1}{0.25} - 1\right| \\ \ln\left|\frac{1}{0.5} - 1\right| \end{bmatrix} \\
\begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \ln\left|\frac{1}{0.2} - 1\right| \\ \ln\left|\frac{1}{0.25} - 1\right| \\ \ln\left|\frac{1}{0.5} - 1\right| \end{bmatrix} \\
\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{7}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \ln\left|\frac{1}{0.2} - 1\right| \\ \ln\left|\frac{1}{0.25} - 1\right| \\ \ln\left|\frac{1}{0.5} - 1\right| \end{bmatrix} \\
\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} -0.6931 \\ 2.2146 \end{bmatrix}
\end{aligned}$$

Step 4:

$$y = \frac{1}{1 + e^{-0.6931x + 2.2146}}$$

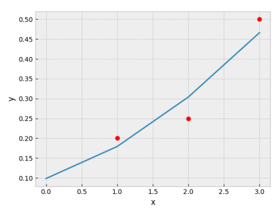

Figure 1: Logistic Regression with 3 data points

Problem 33-2

Solution

(a)

$$\begin{aligned}
\mathrm{E}[aX] &= \int_{-\infty}^{\infty} a \cdot p(x)\, \mathrm{d}x \\
&= a \cdot \int_{-\infty}^{\infty} p(x)\, \mathrm{d}x \\
&= a \cdot \mathrm{E}[X]
\end{aligned}$$

(b)

$$
\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}[(X - \mathrm{E}[X])^2] \\
&= \int_{-\infty}^{\infty} p(x) \cdot (X - E[X])^2 \; \mathrm{d}x \\
&= \int_{-\infty}^{\infty} p(x) \cdot \left(X^2 - 2X\mathrm{E}[X] + \mathrm{E}[X]^2\right) \; \mathrm{d}x \\
&= \int_{-\infty}^{\infty} p(x) \cdot X^2 \, \mathrm{d}x - 2 \int_{-\infty}^{\infty} p(x) \cdot X \cdot \mathrm{E}[X] \, \mathrm{d}x + \int_{-\infty}^{\infty} p(x) \cdot \mathrm{E}[X]^2 \, \mathrm{d}x \\
&= \mathrm{E}[X^2] - 2 \cdot \mathrm{E}[X]^2 + \mathrm{E}[X]^2 \\
&= \mathrm{E}[X^2] - \mathrm{E}[X]^2
\end{aligned}
$$