Computation Modeling Assignment 35

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Problem 35-1

Solution

(a)

$$\begin{aligned} P(1, 17, 8, 25, 3 \,|\, k) &= P(1) + P(17) + P(8) + P(25) + P(3) \\ &= \frac{1}{k} \cdot \frac{1}{k} \cdot \frac{1}{k} \cdot \frac{1}{k} \cdot \frac{1}{k} \\ &= \frac{1}{k^5} \end{aligned}$$

(b)

$$\begin{aligned} \sum_{k=1}^{\infty} c \cdot P(1, 17, 8, 25, 3 \,|\, k) &= 1 \\ c \cdot \sum_{k=1}^{\infty} \frac{1}{k^5} &= 1 \\ c \cdot \sum_{k=25}^{\infty} \frac{1}{k^5} &= 1 \\ c \cdot 0.0000006929 &= 1 \\ c &= 1443199.783177032 \end{aligned}$$

(c)

$$\begin{aligned} P(k \,|\, 1, 17, 8, 25, 3) &= c \cdot P(1, 17, 8, 25, 3 \,|\, k) \\ &= c \cdot \begin{cases} \frac{1}{k^5} & k \geq 25 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{c}{k^5} & k \geq 25 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$P(k \,|\, 1, 17, 8, 25, 3)$ is maximized when k is at its lowest value, which in this case is 25. Hence, the most probable value of k is 25

(d)

$$\begin{aligned}
P(25 \mid 1, 17, 8, 25, 3) &= c \cdot P(1, 17, 8, 25, 3 \mid k) \\
&= 1443199.783177032 \cdot \frac{1}{25^5} \\
&= 0.1477836578
\end{aligned}$$

(e)

$$\begin{aligned}
P(25 \leq k \leq 30 \mid 1, 17, 8, 25, 3) &= \sum_{n=25}^{30} c \cdot P(1, 17, 8, 25, 3 \mid k) \\
&= \sum_{k=25}^{30} 1443199.783177032 \cdot \frac{1}{k^5} \\
&= 1443199.783177032 \cdot \left(\frac{1}{25^5} + \frac{1}{26^5} + \frac{1}{27^5} + \frac{1}{28^5} + \frac{1}{29^5} + \frac{1}{30^5} \right) \\
&= 0.5834392031
\end{aligned}$$

(f) You can be 95% sure that the upper bound is less than 52.

Problem 35-2

Solution

(a) $p(x, y)$ is a valid probability distribution when

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \, \mathrm{d}x \, \mathrm{d}y &= 1 \\
\int_c^d \int_a^b k \, \mathrm{d}x \, \mathrm{d}y &= 1 \\
\int_c^d kx|_a^b \, \mathrm{d}y &= 1 \\
\int_c^d \int_a^b k \, \mathrm{d}x \, \mathrm{d}y &= 1 \\
\int_c^d kb - ka \, \mathrm{d}y &= 1 \\
(kb - ka)(y)|_c^d &= 1 \\
(kb - ka)(d) - (kb - ka)(c) &= 1 \\
kbd - kad - (kbc - kac) &= 1 \\
k(bd - ad - bc + ac) &= 1 \\
k &= \frac{1}{bd - ad - bc + ac}
\end{aligned}$$

(b)

$$
\begin{aligned}
\mathrm{E}[X] &= \int_c^d \int_a^b x \cdot p(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_c^d \int_a^b \frac{x}{bd-ad-bc+ac}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_c^d \left.\frac{x^2}{2\,(bd-ad-bc+ac)}\right|_a^b \mathrm{d}y \\
&= \int_c^d \frac{b^2}{2\,(bd-ad-bc+ac)} - \frac{a^2}{2\,(bd-ad-bc+ac)}\,\mathrm{d}y \\
&= \left. y\left(\frac{b^2}{2\,(bd-ad-bc+ac)} - \frac{a^2}{2\,(bd-ad-bc+ac)}\right)\right|_c^d \\
&= \left(\frac{db^2}{2\,(bd-ad-bc+ac)} - \frac{da^2}{2\,(bd-ad-bc+ac)}\right) - \left(\frac{cb^2}{2\,(bd-ad-bc+ac)} - \frac{ca^2}{2\,(bd-ad-bc+ac)}\right) \\
&= \frac{db^2-da^2-cb^2+ca^2}{2\,(bd-ad-bc+ac)} \\
&= \frac{b+a}{2}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[Y] &= \int_c^d \int_a^b y \cdot p(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_c^d \int_a^b \frac{y}{bd-ad-bc+ac}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_c^d \left.\frac{yx}{bd-ad-bc+ac}\right|_a^b \mathrm{d}y \\
&= \int_c^d \frac{yb}{bd-ad-bc+ac} - \frac{ya}{bd-ad-bc+ac}\,\mathrm{d}y \\
&= \left.\frac{y^2b}{2\,(bd-ad-bc+ac)} - \frac{y^2a}{2\,(bd-ad-bc+ac)}\right|_c^d \\
&= \frac{d^2b}{2\,(bd-ad-bc+ac)} - \frac{d^2a}{2\,(bd-ad-bc+ac)} - \left(\frac{c^2b}{2\,(bd-ad-bc+ac)} - \frac{c^2a}{2\,(bd-ad-bc+ac)}\right) \\
&= \frac{d^2b-d^2a-c^2b+c^2a}{2\,(bd-ad-bc+ac)} \\
&= \frac{d+c}{2}
\end{aligned}
$$

(c) Geometrically, $\mathrm{E}[X]$ represents the midpoint between a and b while $\mathrm{E}[Y]$ represents the midpoint between c and d. Hence, the point $(\,\mathrm{E}[X]\,, \mathrm{E}[Y]\,)$ represents the center of the rectangle bounded by $x=a, x=b, y=c$ and $y=d$.