Eurisko Assignment 19-2

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To say that a random variable N follows a probability distribution $p(n)$ is to say that $P(N = n) = p(n)$. Symbolically, we write $X \sim p$.

The expected value (also known as the mean) of a random variable $N \sim p$ is defined as the weighted sum of possible values, where the weights are given by the probability. In other words, $E[N] = \sum n \cdot p(n)$.

The variance of a random variable is the expected squared deviation from the mean. In other words, $Var[N] = E[(N - E[N])^2]$.

Part 1

(a) Write the probability distribution $p_4(n)$ for getting n heads on 4 coin flips, where the coin is a fair coin (i.e. it lands on heads with probability 0.5).

The probability of getting n heads in f coin flips with a fair coin is $\frac{f!}{n! \cdot (f-n)!} \cdot \frac{1}{2^f}$.

Let $a(n)$ be equal to the probability of getting n heads in 4 coin flips with a fair coin, which is $\frac{4!}{n! \cdot (4-n)!} \cdot \frac{1}{16} = \frac{1.5}{n! \cdot (4-n)!}$. $p_4(n)$ is equal to $[p_4(0), p_4(1), p_4(2), p_4(3), p_4(4)]$, which also equals $[a(0), a(1), a(2), a(3), a(4)]$.

$$a(0) = \frac{1.5}{0! \cdot (4-0)!} = \frac{1.5}{1 \cdot (4)!} = \frac{1.5}{24} = 0.0625$$
$$a(1) = \frac{1.5}{1! \cdot (4-1)!} = \frac{1.5}{1 \cdot (3)!} = \frac{1.5}{6} = 0.25$$
$$a(2) = \frac{1.5}{2! \cdot (4-2)!} = \frac{1.5}{2 \cdot (2)!} = \frac{1.5}{4} = 0.375$$
$$a(3) = \frac{1.5}{3! \cdot (4-3)!} = \frac{1.5}{6 \cdot (1)!} = \frac{1.5}{6} = 0.25$$
$$a(4) = \frac{1.5}{4! \cdot (4-4)!} = \frac{1.5}{24 \cdot (0)!} = \frac{1.5}{24} = 0.0625$$

Therefore, the probability distribution $p_4(n)$ for getting n heads on 4 coin flips, where the coin is a fair coin, is $[0.0625, 0.25, 0.375, 0.25, 0.0625]$.

(b) Let N be the number of heads in 4 coin flips. Then $N \sim p_4$. Intuitively, what is the expected value of N? Explain the reasoning behind your intuition.

The expected value of N is the number of heads that has the highest probability of being flipped. This is biggest number in $p_4(n)$. The biggest number in $p_4(n)$ is 0.375, which is the probability of getting 2 heads in 4 flips. Therefore, the expected value of N is 2.

(c) Compute the expected value of N, using the definition $E[N] = \sum n \cdot p(n)$. The answer you get should match your answer from (b).

$E[N] = \sum n \cdot p(n) = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4) = 0 + 1 \cdot 0.25 + 2 \cdot 0.375 + 3 \cdot 0.25 + 4 \cdot 0.0625 = 0 + 0.25 + 0.75 + 0.75 + 0.25 = 2$

(d) Compute the variance of N, using the definition $Var[N] = E[(N - E[N])^2]$. Your answer should come out to 1.

$Var[N] = E[(N - E[N])^2] = E[(N-2)^2] = E[4, 1, 0, 1, 4] = \sum n \cdot p(n) = 4 \cdot p(4) + 1 \cdot p(1) + 0 \cdot p(0) + 1 \cdot p(1) + 4 \cdot p(4) = 4 \cdot 0.0625 + 1 \cdot 0.25 + 0 + 1 \cdot 0.25 + 4 \cdot 0.0625 = 0.25 + 0.25 + 0.25 + 0.25 = 1$

Part 2

(a) Write the probability distribution $p_{4,k}(n)$ for getting n heads on 4 coin flips, where the coin is a biased coin that lands on heads with probability k. If you substitute $k = 0.5$, you should get the same result that you did in Part 1.a.

The probability of getting n in f coin flips with a biased coin is $\frac{f!}{n! \cdot (f-n)!} \cdot k^n \cdot (1-k)^{f-n}$.

Let $b(n)$ be equal to the probability of getting n heads in 4 coin flips with a biased coin, which is $\frac{24}{n! \cdot (4-n)!} \cdot k^n \cdot (1-k)^{4-n}$. $p_{4,k}(n)$ is equal to $[p_{4,k}(0), p_{4,k}(1), p_{4,k}(2), p_{4,k}(3), p_{4,k}(4)]$, which also equals $[b(0), b(1), b(2), b(3), b(4)]$.

$$b(0) = \frac{24}{0! \cdot (4-0)!} \cdot k^0 \cdot (1-k)^{4-0} = \frac{24}{1 \cdot 4!} \cdot 1 \cdot (1-k)^4 = \frac{24}{24} \cdot (1-k)^4 = (1-k)^4$$
$$b(1) = \frac{24}{1! \cdot (4-1)!} \cdot k^1 \cdot (1-k)^{4-1} = \frac{24}{1 \cdot 3!} \cdot k^1 \cdot (1-k)^3 = \frac{24}{6} \cdot k^1 \cdot (1-k)^3 = 4k(1-k)^3$$
$$b(2) = \frac{24}{2! \cdot (4-2)!} \cdot k^2 \cdot (1-k)^{4-2} = \frac{24}{4 \cdot 2!} \cdot k^2 \cdot (1-k)^2 = \frac{24}{4} \cdot k^2 \cdot (1-k)^2 = 6k^2(1-k)^2$$
$$b(3) = \frac{24}{3! \cdot (4-3)!} \cdot k^3 \cdot (1-k)^{4-3} = \frac{24}{6 \cdot 1!} \cdot k^3 \cdot (1-k)^1 = \frac{24}{6} \cdot k^3 \cdot (1-k)^1 = 4k^3(1-k)$$
$$b(4) = \frac{24}{4! \cdot (4-4)!} \cdot k^4 \cdot (1-k)^{4-4} = \frac{24}{24 \cdot 0!} \cdot k^4 \cdot (1-k)^0 = \frac{24}{24} \cdot k^4 \cdot 1 = 1 \cdot k^4 \cdot 1 = k^4$$

Therefore, the probability distribution $p_{4,k}(n)$ for getting n heads on 4 coin flips, where the coin is a biased coin, is $[(1-k)^4, 4k(1-k)^3, 6k^2(1-k)^2, 4k^3(1-k), k^4]$.

When we try $k = 0.5$, we find that $p_{4,0.5}(n) = [(1-0.5)^4, 4 \cdot 0.5 \cdot (1-0.5)^3, 6 \cdot (0.5)^2 \cdot (1-0.5)^2, 4 \cdot (0.5)^3 \cdot (1-0.5), (0.5)^4] = [(0.5)^4, 2 \cdot (0.5)^3, 6 \cdot (0.5)^2 \cdot (0.5)^2, 4 \cdot (0.5)^3 \cdot (0.5), (0.5)^4] = [0.0625, 2 \cdot 0.125, 6 \cdot 0.25 \cdot 0.25, 4 \cdot 0.125 \cdot 0.5, 0.0625] =$

$[0.0625, 0.25, 0.375, 0.25, 0.0625]$. This was what we got for Part 1.a, so we know that our $p_{4,k}(n)$ is correct.

(b) Let N be the number of heads in 4 coin flips of a biased coin. Then $N \sim p_{4,k}$. Intuitively, what is the expected value of N? Your answer should be in terms of k. Explain the reasoning behind your intuition. If you substitute $k = 0.5$, you should get the same result that you did in Part 1.b.

The expected value of N is the number of heads that has the highest probability of being flipped in $p_{4,k}(n)$. This number is whatever the bias of the coin is. If the coin has a bias of 1, then the excepted value of N will be 4. If the coin has a bias of 0, the expected value of N will be 0. Because any number in $p_{4,k}(n)$ is between 0 and 1, and we want to get a number that is between 0 and 4, we can conclude that the expected value of N is $4 \cdot k$.

(c) Compute the expected value of N, using the definition $E[N] = \sum n \cdot p(n)$. The answer you get should match your answer from 2.b.

$E[N] = \sum n \cdot p(n) = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + 4 \cdot p(4) = 0 \cdot (1-k)^4 + 1 \cdot 4k(1-k)^3 + 2 \cdot 6k^2(1-k)^2 + 3 \cdot 4k^3(1-k) + 4 \cdot k^4 = 4k(1-k)^3 + 12k^2(1-k)^2 + 12k^3(1-k) + 4k^4$

When we multiply $4k(1-k)^3 + 12k^2(1-k)^2 + 12k^3(1-k) + 4k^4$ out, then simplify, we get $4k$.