Eurisko Assignment 20-1

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Continuous distributions are defined similarly to discrete distributions. There are only 2 big differences:

1. We use an integral to compute expectation: if $X \sim p$, then

$$E[X] = \int_{-\infty}^{\infty} x \cdot p(x)\, dx$$

2. We talk about probability on an interval rather than at a point: if $X \sim p$, then

$$P(a < X \leq b) = \int_a^b p(x)\, dx$$

Problems

Consider the exponential distribution defined by

$$p_2(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) Using integration, show that this is a valid distribution, i.e. all the probability integrates to 1.

$$\begin{aligned} \int_{-\infty}^{\infty} p_2(x)\, dx &= \int_{-\infty}^{0} 0\, dx + \int_0^{\infty} 2e^{-2x}\, dx \\ &= \int_0^{\infty} 2e^{-2x}\, dx \\ &= -e^{-2x}\Big|_0^{\infty} \\ &= \Big(-e^{-2\cdot\infty}\Big) - \Big(-e^{-2\cdot 0}\Big) \\ &= \Big(-e^{-\infty}\Big) - \Big(-e^{0}\Big) \end{aligned}$$

$$= (0) - (-1)$$
$$= 1$$

(b) Given that $X \sim p_2$, compute $P(0 < X \leq 1)$. You should get a result of $1 - e_2$.

$$\begin{aligned}
P(0 < X \leq 1) &= \int_0^1 p_2(x)\,dx \\
&= \int_0^1 2e^{-2x}\,dx \\
&= -e^{-2x}\Big|_0^1 \\
&= \Big(-e^{-2}\Big) - \Big(-e^{0}\Big) \\
&= -e^{-2} - -1 \\
&= -e^{-2} + 1 \\
&= 1 - e^{-2}
\end{aligned}$$

(c) Given that $X \sim p_2$, compute $E[X]$. You should get a result of $\frac{1}{2}$.

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x \cdot p(x)\,dx \\
&= \int_{-\infty}^{0} x \cdot 0\,dx + \int_0^{\infty} x \cdot 2e^{-2x}\,dx \\
&= \int_0^{\infty} 2x \cdot e^{-2x}\,dx \\
&= 2x \cdot -\frac{e^{-2x}}{2}\Big|_0^{\infty} - \int_0^{\infty} -\frac{e^{-2x}}{2} \cdot 2\,dx \\
&= -xe^{-2x}\Big|_0^{\infty} + \int_0^{\infty} e^{-2x}\,dx \\
&= \bigg(\Big(-\infty \cdot e^{-\infty}\Big) - \Big(-0 \cdot e^{0}\Big)\bigg) - \frac{e^{-2x}}{2}\Big|_0^{\infty} \\
&= \Big(-\infty \cdot 0\Big) - 0 - \bigg(\Big(\frac{e^{-\infty}}{2}\Big) - \Big(\frac{e^{0}}{2}\Big)\bigg) \\
&= 0 - \left(\frac{0}{2} - \frac{1}{2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

(d) Given that $X \sim p_2$, compute $Var[X]$. You should get a result of $\frac{1}{4}$.

$$
\begin{aligned}
Var[X] &= E\left[\left(X - E[X]\right)^2\right] \\
&= E\left[\left(X - \frac{1}{2}\right)^2\right] \\
&= \int_{-\infty}^{\infty} \left(x - \frac{1}{2}\right)^2 \cdot p(x)\, dx \\
&= \int_{-\infty}^{0} \left(x - \frac{1}{2}\right)^2 \cdot 0\, dx + \int_{0}^{\infty} \left(x - \frac{1}{2}\right)^2 \cdot 2e^{-2x}\, dx \\
&= \int_{0}^{\infty} \left(x - \frac{1}{2}\right)^2 \cdot 2e^{-2x}\, dx \\
&= \left(\left(x - \frac{1}{2}\right)^2 \cdot \left(-e^{-2x}\right)\right)\Bigg|_0^\infty - \int_0^\infty \left(-e^{-2x}\right) \cdot (2x-1)\, dx \\
&= \left(\left(\infty - \frac{1}{2}\right)^2 \cdot \left(-e^{-\infty}\right)\right) - \left(\left(0 - \frac{1}{2}\right)^2 \cdot \left(-e^{0}\right)\right) + (2x-1)\left(-\frac{e^{-2x}}{2}\right)\Bigg|_0^\infty + \int_0^\infty e^{-2x}\, dx \\
&= \left(\infty \cdot 0\right) - \left(\frac{1}{4} \cdot -1\right) - \left(xe^{-2x} - \frac{e^{-2x}}{2}\right)\Bigg|_0^\infty - \left(\frac{e^{-2x}}{2}\right)\Bigg|_0^\infty \\
&= 0 + \frac{1}{4} - \left(xe^{-2x}\right)\Bigg|_0^\infty \\
&= \frac{1}{4} - \left(\left(\infty \cdot e^{-\infty}\right) - \left(0 \cdot e^{0}\right)\right) \\
&= \frac{1}{4} - \left((\infty \cdot 0) - 0\right) \\
&= \frac{1}{4}
\end{aligned}
$$