Eurisko Assignment 21-2

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A uniform distribution U on the interval $[3, 7]$ is a probability distribution $p(x)$ that takes the following form for some constant k:

$$p(x) = \begin{cases} k & x \in [3, 7] \\ 0 & x \notin [3, 7] \end{cases}$$

It is also $U[3, 7]$. So, to say that $X \sim U[3, 7]$, is to say that $X \sim p$ for the function p shown above.

We use an integral to compute expectation: if $X \sim p$, then

$$E[X] = \int_{-\infty}^{\infty} x \cdot p(x)\, dx$$

Also, the variance of a variable $Var[N]$ equals

$$Var[N] = E\left[\Big(N - E[N]\Big)^2\right].$$

(a) Find the value of k such that $p(x)$ is a valid probability distribution. (Remember that for a function to be a valid probability distribution, it must integrate to 1.)

Since the probability distribution is uniform, the probability of getting a $3, 4, 5, 6$, or 7 are equal. Since the sum of every probability equals 1, and the probability of getting each number is equal,

$$
\begin{aligned}
\int_3^7 p(x)\,dx &= 1 \\
\int_3^7 k\,dx &= 1 \\
k \cdot \int_3^7 1\,dx &= 1 \\
k \cdot x \Big|_3^7 &= 1 \\
k \cdot (7-3) &= 1 \\
4k &= 1 \\
k &= 0.25.
\end{aligned}
$$

(b) Given that $X \sim U[3,7]$, compute $E[X]$. Check: does your result make intuitive sense? If you pick a bunch of numbers from the interval $[3,7]$ and all of those numbers are equally likely choices, then what would you expect to be the average of the numbers you pick?

$$
\begin{aligned}
E[X] &= \int_3^7 x \cdot p(x)\,dx \\
&= \int_3^7 x \cdot k\,dx \\
&= k \cdot \int_3^7 x\,dx \\
&= 0.25 \cdot \frac{x^2}{2}\Big|_3^7 \\
&= 0.25 \cdot \left(\frac{7^2}{2} - \frac{3^2}{2}\right) \\
&= 0.25 \cdot (24.5 - 4.5) \\
&= 0.25 \cdot 20 \\
E[X] &= 5
\end{aligned}
$$

This makes intuitive sense. If you take pick a random number from a bunch of numbers with even probabilities, the average outcome will be the average number in the bunch. The average of $3, 4, 5, 6,$ and 7 is

$$
\frac{3+4+5+6+7}{5} = \frac{25}{5} = 5
$$

(b) Given that $X \sim U[3,7]$, compute $Var[X]$. You should get $\frac{4}{3}$.

$$
\begin{aligned}
Var[N] &= E\left[\left(N - E[N]\right)^2\right] \\
&= E\left[(N-5)^2\right] \\
&= \int_3^7 (x-5)^2 \cdot p(x)\, dx \\
&= \int_3^7 (x-5)^2 \cdot 0.25\, dx \\
&= 0.25 \cdot \int_3^7 (x-5)^2\, dx \\
&= 0.25 \cdot \left.\frac{(x-5)^3}{3}\right|_3^7 \\
&= 0.25 \cdot \left(\frac{(7-5)^3}{3} - \frac{(3-5)^3}{3}\right) \\
&= 0.25 \cdot \left(\frac{2^3}{3} - \frac{(-2)^3}{3}\right) \\
&= 0.25 \cdot \left(\frac{8}{3} - \frac{-8}{3}\right) \\
&= 0.25 \cdot \frac{16}{3} \\
Var[N] &= \frac{4}{3}
\end{aligned}
$$