Eurisko Assignment 23-2

Charlie Weinberger

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Reminders:

1. We use an integral to compute expectation: if $X \sim p$, then

$$E[X] = \int_{-\infty}^{\infty} x \cdot p(x)\,\mathrm{d}x$$

2. The variance of a random variable is the expected squared deviation from the mean: if $X \sim p$, then

$$Var[X] = E\left[\left(X - E[X]\right)^2\right]$$

3. We talk about probability on an interval rather than at a point: if $X \sim p$, then

$$P(a < X \leq b) = \int_a^b p(x)\,\mathrm{d}x$$

Part 1

Consider the exponential distribution defined by

$$p_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) Using integration, show that this is a valid distribution, i.e. all the probability integrates to 1.

$$\begin{aligned}
\int_{-\infty}^{\infty} p_\lambda(x)\,\mathrm{d}x &= \int_{-\infty}^{0} 0\,\mathrm{d}x + \int_{0}^{\infty} \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= \int_{0}^{\infty} \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= -e^{-\lambda x}\Big|_{0}^{\infty} \\
&= \Big(-e^{-\infty}\Big) - \Big(-e^{0}\Big) \\
&= (0) - (-1) \\
&= 1
\end{aligned}$$

(b) Given that $X \sim p_\lambda$, compute $P(0 < X < 1)$.

$$\begin{aligned}
\mathrm{P}(0 < X \leq 1) &= \int_{0}^{1} p_\lambda(x)\,\mathrm{d}x \\
&= \int_{0}^{1} \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= -e^{-\lambda x}\Big|_{0}^{1} \\
&= \Big(-e^{-\lambda}\Big) - \Big(-e^{0}\Big) \\
&= -e^{-\lambda} - (-1) \\
&= -e^{-\lambda} + 1 \\
&= 1 - e^{-\lambda}
\end{aligned}$$

(c) Given that $X \sim p_\lambda$, compute $\mathrm{E}[X]$.

$$\begin{aligned}
\mathrm{E}[X] &= \int_{-\infty}^{\infty} x \cdot p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{0} x \cdot 0\,\mathrm{d}x + \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= \int_{0}^{\infty} \lambda x \cdot e^{-\lambda x}\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \lambda x \cdot -\frac{e^{-\lambda x}}{\lambda}\bigg|_0^\infty - \int_0^\infty -\frac{e^{-\lambda x}}{\lambda} \cdot \lambda \,\mathrm{d}x \\
&= -xe^{-\lambda x}\bigg|_0^\infty + \int_0^\infty e^{-\lambda x}\,\mathrm{d}x \\
&= \bigg(\Big(-\infty \cdot e^{-\infty}\Big) - \Big(-0 \cdot e^0\Big)\bigg) - \frac{e^{-\lambda x}}{\lambda}\bigg|_0^\infty \\
&= \Big(-\infty \cdot 0\Big) - 0 - \left(\left(\frac{e^{-\infty}}{\lambda}\right) - \left(\frac{e^0}{\lambda}\right)\right) \\
&= 0 - \left(\frac{0}{\lambda} - \frac{1}{\lambda}\right) \\
&= \frac{1}{\lambda}
\end{aligned}
$$

(d) Given that $X \sim p_\lambda$, compute $\mathrm{Var}[X]$.

$$
\begin{aligned}
\mathrm{Var}[X] &= E\bigg[\Big(X - E[X]\Big)^2\bigg] \\
&= E\left[\left(X - \frac{1}{\lambda}\right)^2\right] \\
&= \int_{-\infty}^\infty \left(x - \frac{1}{\lambda}\right)^2 \cdot p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^0 \left(x - \frac{1}{\lambda}\right)^2 \cdot 0\,\mathrm{d}x + \int_0^\infty \left(x - \frac{1}{\lambda}\right)^2 \cdot \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= \int_0^\infty \left(x - \frac{1}{\lambda}\right)^2 \cdot \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= \left(\left(x - \frac{1}{\lambda}\right)^2 \cdot \Big(-e^{-\lambda x}\Big)\right)\bigg|_0^\infty - \int_0^\infty \Big(-e^{-\lambda x}\Big) \cdot (\lambda x - 1)\,\mathrm{d}x \\
&= \left(\left(\infty - \frac{1}{\lambda}\right)^2 \cdot \Big(-e^{-\infty}\Big)\right) - \left(\left(0 - \frac{1}{\lambda}\right)^2 \cdot \Big(-e^0\Big)\right) + (\lambda x - 1)\left(-\frac{e^{-\lambda x}}{\lambda}\right)\bigg|_0^\infty + \int_0^\infty e^{-\lambda x}\,\mathrm{d}x \\
&= \Big(\infty \cdot 0\Big) - \left(\frac{1}{\lambda^2} \cdot -1\right) - \left(xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}\right)\bigg|_0^\infty - \left(\frac{e^{-\lambda x}}{\lambda}\right)\bigg|_0^\infty \\
&= 0 + \frac{1}{\lambda^2} - \Big(xe^{-\lambda x}\Big)\bigg|_0^\infty
\end{aligned}
$$

$$= \frac{1}{\lambda^2} - \left(\left(\infty \cdot e^{-\infty} \right) - \left(0 \cdot e^0 \right) \right)$$
$$= \frac{1}{\lambda^2} - \left((\infty \cdot 0) - 0 \right)$$
$$= \frac{1}{\lambda^2}$$

Part 2

Consider the general uniform distribution on the interval $[a, b]$. It takes the following form for some constant k:

$$p_\lambda(x) = \begin{cases} k & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$

(a) Find the value of k such that $p(x)$ is a valid probability distribution. Your answer should be in terms of a and b.

$$\begin{aligned} \int_a^b p(x)\,\mathrm{d}x &= 1 \\ \int_a^b k\,\mathrm{d}x &= 1 \\ k \cdot \int_a^b 1\,\mathrm{d}x &= 1 \\ k \cdot x \Big|_a^b &= 1 \\ k \cdot (b - a) &= 1 \\ k &= \frac{1}{b - a} \end{aligned}$$

(b) Given that $X \sim p$, compute the cumulative distribution $P(X \leq x)$. Your answer should be a piecewise function:

$$P(X \leq x) = \begin{cases} & x < a \\ & a \leq x \leq b \\ & b < x \end{cases}$$

For $x < a$:

$$\int_{-\infty}^{a} p_\lambda(x)\,\mathrm{d}x = \int_{-\infty}^{a} 0\,\mathrm{d}x = 0$$

For $a \leq x \leq b$:

$$\begin{aligned}
\int_{a}^{b} p_\lambda(x)\,\mathrm{d}x &= \int_{a}^{b} k\,\mathrm{d}x \\
&= \int_{a}^{b} \frac{1}{b-a}\,\mathrm{d}x \\
&= \frac{1}{b-a} \cdot \int_{a}^{b} 1\,\mathrm{d}x \\
&= \frac{1}{b-a} \cdot x\Big|_{a}^{b} \\
&= \frac{1}{b-a} \cdot (b-a) \\
&= 1
\end{aligned}$$

For $b < x$:

$$\int_{b}^{\infty} p_\lambda(x)\,\mathrm{d}x = \int_{b}^{\infty} 0\,\mathrm{d}x = 0$$

Therefore,

$$P(X \leq x) = \begin{cases} 0 & x < a \\ 1 & a \leq x \leq b \\ 0 & b < x \end{cases}$$

(c) Given that $X \sim p$, compute $\mathrm{E}[X]$.

$$\begin{aligned}
\mathrm{E}[X] &= \int_a^b x \cdot p(x)\,\mathrm{d}x \\
&= \int_a^b x \cdot k\,\mathrm{d}x \\
&= k \cdot \int_a^b x\,\mathrm{d}x \\
&= \frac{1}{b-a} \cdot \frac{x^2}{2}\bigg|_a^b \\
&= \frac{1}{b-a} \cdot \left(\frac{b^2}{2} - \frac{a^2}{2}\right) \\
&= \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} \\
&= \frac{1}{b-a} \cdot \frac{(b-a)(b+a)}{2} \\
\mathrm{E}[X] &= \frac{a+b}{2}
\end{aligned}$$

(d) Given that $X \sim p$, compute $\mathrm{Var}[X]$.

$$\begin{aligned}
\mathrm{Var}[X] &= E\left[\Big(X - E[X]\Big)^2\right] \\
&= E\left[\left(X - \frac{a+b}{2}\right)^2\right] \\
&= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \cdot p(x)\,\mathrm{d}x \\
&= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \cdot k\,\mathrm{d}x \\
&= \int_a^b \left(x - \frac{a+b}{2}\right)^2 \cdot \frac{1}{b-a}\,\mathrm{d}x \\
&= \frac{1}{b-a} \cdot \int_a^b \left(x - \frac{a+b}{2}\right)^2\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b-a} \cdot \frac{\left(x - \frac{a+b}{2}\right)^3}{3} \Bigg|_a^b \\
&= \frac{1}{3 \cdot (b-a)} \cdot \left(x - \frac{a+b}{2}\right)^3 \Bigg|_a^b \\
&= \frac{1}{3 \cdot (b-a)} \cdot \left(\left(b - \frac{a+b}{2}\right)^3 - \left(a - \frac{a+b}{2}\right)^3\right) \\
&= \frac{1}{3 \cdot (b-a)} \cdot \left(\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right) \\
&= \frac{1}{3 \cdot (b-a)} \cdot \left(\frac{(b-a)^3}{8} - \frac{(a-b)^3}{8}\right) \\
&= \frac{1}{24 \cdot (b-a)} \cdot \left((b-a)^3 - (a-b)^3\right) \\
&= \frac{(b-a)^3 - (a-b)^3}{24 \cdot (b-a)} \\
&= \frac{2 \cdot (b-a)^3}{24 \cdot (b-a)} \\
\mathrm{Var}[X] &= \frac{(b-a)^2}{12}
\end{aligned}
$$