Eurisko Assignment 23-2

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Part 1

Suppose that you take a bus to work every day. Bus A arrives at 8am but is x minutes late with $x \sim U(0, 20)$. Bus B arrives at 8:10 but with $x \sim U(0, 10)$. The bus ride is 20 minutes and you need to arrive at work by 8:30.

Remember that $U(a, b)$ means the uniform distribution on $[a, b]$. See problem 23-2 if you need a refresher on exponential distributions.

Recall the formulas for the mean and variance of uniform distributions: If $X \sim U(a, b)$, then $E[X] = \dfrac{a+b}{2}$ and $Var(X) = \dfrac{(b-a)^2}{12}$. You can use these formulas without any further justification.

(a) If you take bus A, what time do you expect to arrive at work? Justify your answer.

The expected time to get to work is 8:00 am plus E[x] minutes plus 20 minutes, or:

$$\begin{aligned} 8{:}00 + E[x] + 20 &= 8{:}00 + \frac{0+20}{2} + 20 \\ &= 8{:}00 + 10 + 20 \\ &= 8{:}30 \end{aligned}$$

Therefore, you can expect to be at work at 8:30.

(b) If you take bus B, what time do you expect to arrive at work? Justify your answer.

The expected time to get to work is 8:10 am plus E[x] minutes plus 20 minutes, or:

$$
\begin{aligned}
&8{:}10 + E[x] + 20 \\
&= 8{:}10 + \frac{0+10}{2} + 20 \\
&= 8{:}10 + 5 + 20 \\
&= 8{:}35
\end{aligned}
$$

Therefore, you can expect to be at work at 8:35.

(c) If you take bus A, what is the probability that you will arrive on time to work? Justify your answer.

The probability that you will arrive at work on time is the probability that 8:00 am plus x minutes plus 20 minutes is less than or equal to 8:30. That is,

$$
\begin{aligned}
8{:}00 + x + 20 &\leq 8{:}30 \\
8{:}20 + x &\leq 8{:}30 \\
x &\leq 10
\end{aligned}
$$

That means that the probability of you arriving at work on time is the probability that $x \leq 10$. The probability of that is:

$$
\begin{aligned}
\int_0^{10} p(x)\mathrm{d}x &= \int_0^{10} \frac{1}{20}\mathrm{d}x \\
&= \left.\frac{x}{20}\right|_0^{10} \\
&= \frac{10}{20} - \frac{0}{20} \\
&= 0.5
\end{aligned}
$$

This tells us that the probability of you arriving at work on time is 0.5, which means that you will arrive on time to work 50 percent of the time when you take bus A.

(d) If you take bus B, what is the probability that you will arrive on time to work? Justify your answer.

The probability that you will arrive at work on time is the probability that 8:10 am plus x minutes plus 20 minutes is less than or equal to 8:30. That is,

$$\begin{aligned} 8{:}10 + x + 20 &\leq 8{:}30 \\ 8{:}30 + x &\leq 8{:}30 \\ x &= 0 \end{aligned}$$

That means that the probability of you arriving at work on time is the probability that $x = 10$. The probability of that is:

$$\int_0^0 p(x)\mathrm{d}x = 0$$

This tells us that the probability of you arriving at work on time is 0, which means that you will never arrive on time to work on time when you take bus B.

Part 2

Continuing the scenario above, there is a third option that you can use to get to work: you can jump into a wormhole and (usually) come out almost instantly at the other side. The only issue is that time runs differently inside the wormhole, and while you're probably going to arrive at the other end very quickly, there's a small chance that you could get stuck in there for a really long time.

The number of seconds it takes you to come out the other end of the wormhole follows an exponential distribution $\mathrm{Exp}(\lambda = 4) = 4e^{-4x}$.

Recall the formulas for the mean and variance of exponential distributions: If $X \sim \mathrm{Exp}(\lambda)$, then $\mathrm{E}[X] = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$. You can use these formulas without any further justification.

(a) How long do you expect it to take you to come out of the wormhole? Justify your answer.

The expected number of seconds it takes to travel through the wormhole is $\mathrm{E}[X]$, or $\frac{1}{\lambda}$. Since $\lambda = 4$, $\mathrm{E}[X] = \frac{1}{4}$. Therefore, I expect it to take 0.25 seconds to come out of the answer.

(b) What's the probability of taking longer than a second to come out of the wormhole? Justify your answer.

$$\begin{aligned}
\int_1^\infty p(x)\mathrm{d}x &= \int_1^\infty 4e^{-4x}\mathrm{d}x \\
&= -e^{-4x}\Big|_1^\infty \\
&= \Big(-e^{-\infty}\Big) - \Big(-e^{-4}\Big) \\
&= e^{-4}
\end{aligned}$$

(c) Fill in the blank: the probability of coming out of the wormhole within () seconds is 99.999 percent. Justify your answer.

$$\begin{aligned}
\int_0^s 4e^{-4x}\mathrm{d}x &= 0.99999 \\
-e^{-4x}\Big|_0^s &= 0.99999 \\
s &= 2.87823137
\end{aligned}$$

(d) Your friend says that you shouldn't use the wormhole because there's always a chance that you might get stuck in it for over a day, and if you use the wormhole often, then that'll probably happen sometime within your lifetime. Is this a reasonable fear? Why or why not? Justify your answer by computing the probability that you'll get stuck in the wormhole for over a day if you use the wormhole 10 times each day for 80 years.

Hint: It's easier to start by computing the probability that you won't get stuck in the wormhole for over a day on any given trip through the wormhole, and then use that to compute the probability that you won't get stuck in the wormhole for over a day if you use the wormhole 10 times each day for 80 years.

The number of seconds in a day is 86,400 seconds. The probability that you won't get stuck in the wormhole in a day is:

$$\begin{aligned}
\int_0^{86400} p(x)\mathrm{d}x &= \int_0^{86400} 4e^{-4x}\mathrm{d}x \\
&= -e^{-4x}\Big|_0^8 6400 \\
&= \Big(-e^{-345600}\Big) - \Big(-e^{0}\Big) \\
&= 0 - (-1) \\
&= 1
\end{aligned}$$

This means that the probability that you won't get stuck in a wormhole in a day is 100 percent. Realistically, looking at our computations, we realize that while $e^{-345600}$ is not equal to zero, it is equal to a very small number.

Multiplying the 0 percent change of getting stuck in a wormhole by 10 times a day, by 365 days a year, and by 80 years, we still get 0 percent, so there is a 0 percent (or a very low percent) change of getting stuck.