Eurisko Assignment 27-3

Charlie Weinberger

March 2, 2021

Suppose you toss a coin 10 times and get the result $HHHHTHHHHH$. From this result, you estimate that the coin is biased and generally lands on heads 90% of the time. But how sure can you be? Let's quantify it.

Remember that $P(\mathrm{H} \mid k) = k$ and that $P(\mathrm{T} \mid k) = 1 - k$.

(a) Compute the likelihood $P(\mathrm{HHHHT\ HHHHH} \mid k)$ where $P(H) = k$. Remember that the likelihood is just the probability of getting the result $HHHHTHHHHH$ under the assumption that $P(H) = k$. Your answer should be expressed in terms of k.

$$\begin{aligned}
P(\mathrm{HHHHT\ HHHHH} \mid k) &= P(\mathrm{H} \mid k) \cdot P(\mathrm{H} \mid k) \cdot P(\mathrm{H} \mid k) \cdot P(\mathrm{H} \mid k) \cdot P(\mathrm{H} \mid k) \cdot P(\mathrm{H} \mid k) \cdot P(\mathrm{H} \mid k) \\
&\quad \cdot P(\mathrm{H} \mid k) \cdot P(\mathrm{H} \mid k) \cdot P(\mathrm{T} \mid k) \\
&= \Big(P(\mathrm{H} \mid k)\Big)^9 \cdot P(\mathrm{T} \mid k) \\
&= k^9 \cdot (1 - k)
\end{aligned}$$

(b) The likelihood $P(\mathrm{HHHHT\ HHHHH} \mid k)$ can almost be interpreted as a probability distribution for k. The only problem is that it doesn't integrate to 1.

Create a probability distribution $P(k \mid \mathrm{HHHHT\ HHHHH})$ that is proportional to the likelihood $P(\mathrm{HHHHT\ HHHHH} \mid k)$. In other words, find the function $P(k)$ such that

$$P(k \mid \mathrm{HHHHT\ HHHHH}) = c \cdot P(\mathrm{HHHHT\ HHHHH} \mid k)$$

for some constant c, and

$$\int_0^1 P(k \mid \mathrm{HHHHT\ HHHHH}) = 1$$

Note: the distribution $P(k \mid \text{HHHHT HHHHH})$ is called the "posterior" distribution because it represents the probability distribution of k after we have observed the event $HHHHTHHHHH$. The probability distribution of k before we observed the event is called the "prior" distribution and in this case was given by $P(k) \sim U[0,1]$ since we did not know anything about whether or not the coin is biased (or how biased it is).

Answer:

We know that

$$\begin{aligned}
P(k \mid \text{HHHHT HHHHH}) &= c \cdot P(\text{HHHHT HHHHH} \mid k) \\
&= c \cdot k^9 \cdot (1-k) \\
&= c \cdot \left(k^9 - k^{10}\right)
\end{aligned}$$

We can substitute $c \cdot \left(k^9 - k^{10}\right)$ for $P(k \mid \text{HHHHT HHHHH})$ to get

$$\begin{aligned}
1 &= \int_0^1 P(k \mid \text{HHHHT HHHHH}) \\
&= \int_0^1 c \cdot \left(k^9 - k^{10}\right) \mathrm{d}k \\
&= c \cdot \left(\frac{k^{10}}{10} - \frac{k^{11}}{11}\right)\Bigg|_0^1 \\
&= c \cdot \left(\left(\frac{1^{10}}{10} - \frac{1^{11}}{11}\right) - \left(\frac{0^{10}}{10} - \frac{0^{11}}{11}\right)\right) \\
&= c \cdot \left(\frac{1}{10} - \frac{1}{11}\right) \\
1 &= c \cdot 0.00909090909 \\
c &= 110
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(k \mid \text{HHHHT HHHHH}) &= 110 \cdot P(\text{HHHHT HHHHH} \mid k) \\
&= 110 \cdot k^9 \cdot (1-k) \\
&= 110 \cdot (k^9 - k^{10})
\end{aligned}$$

(c) Using the prior distribution $P(k) \sim U[0,1]$, what was the prior probability that the coin was biased towards heads? In other words, what was $P(k > 0.5)$?

$$\begin{aligned}
P(k > 0.5) &= \int_{0.5}^{1} P(k)\mathrm{d}k \\
&= \int_{0.5}^{1} \frac{1}{1-0}\mathrm{d}k \\
&= \int_{0.5}^{1} \mathrm{d}k \\
&= x\Big|_{0.5}^{1} \\
&= 1 - 0.5 \\
P(k > 0.5) &= 0.5
\end{aligned}$$

(d) Using the posterior distribution $P(k \,|\, \text{HHHHT HHHHH})$, what was the posterior probability that the coin was biased towards heads? In other words, what is $P(k > 0.5 \,|\, \text{HHHHT HHHHH})$.

$$\begin{aligned}
P(k > 0.5 \,|\, \text{HHHHT HHHHH}) &= \int_{0.5}^{1} P(k \,|\, \text{HHHHT HHHHH})\mathrm{d}k \\
&= \int_{0.5}^{1} 110 \cdot (k^9 - k^{10})\mathrm{d}k \\
&= 110 \cdot \left(\frac{k^{10}}{10} - \frac{k^{11}}{11}\right)\Bigg|_{0.5}^{1} \\
&= 110 \cdot \left(\left(\frac{1^{10}}{10} - \frac{1^{11}}{11}\right) - \left(\frac{(0.5)^{10}}{10} - \frac{(0.5)^{11}}{11}\right)\right) \\
&= 110 \cdot 0.00903764205 \\
P(k > 0.5 \,|\, \text{HHHHT HHHHH}) &= 0.994140625
\end{aligned}$$

(e) Compare your answers in parts (c) and (d). Did the probability that the coin was biased towards heads increase or decrease, after observing the sequence of flips? Why does this make intuitive sense?

After observing the sequence of flips, the probability the the coin was biased increased. This is because there are significantly more heads than tails in $HHHHTHHHHH$.

(f) Using the posterior distribution, what is the most probable value of k? In other words, what is value of k at which $P(k \mid \text{HHHHT HHHHH})$ reaches a maximum? Show your work using the first or second derivative test.

Let $f(x) = P(k \mid \text{HHHHT HHHHH}) = 110 \cdot k^9 - 110 \cdot k^{10}$. Then $f'(x) = 990k^8 - 1100k^9$. We find the maximum of $f(x)$ when $f'(x) = 0$. So,

$$\begin{aligned} 990k^8 - 1100k^9 &= 0 \\ k^8 \cdot (990 - 1100k) &= 0 \end{aligned}$$

This means that k reaches a maximum at

$$\begin{aligned} 990 - 1100k &= 0 \\ 1100k &= 990 \\ k &= 0.9 \end{aligned}$$

Therefore, $P(k \mid \text{HHHHT HHHHH})$ reaches a maximum at $k = 0.9$.

(g) Why does your answer to (f) make sense? What's the intuition here?

This makes sense because there are 9 heads out of 10 flips in $HHHHTHHHHH$. Therefore, it makes sense that $P(k \mid \text{HHHHT HHHHH})$, which is the probability of k given the flips $HHHHTHHHHH$, is 9 out of 10, or $k = 0.9$.

(h) What is the probability that the bias k lies within 0.05 (5%) of your answer to part (g)? In other words, what is the probability that $0.85 < k < 0.95$?

$$\begin{aligned} P(0.85 < k < 0.95) &= \int_{0.85}^{0.95} p(k)\,\mathrm{d}k \\ &= \int_{0.85}^{0.95} 110 \cdot (k^9 - k^{10})\,\mathrm{d}k \\ &= 110 \cdot \left(\frac{k^{10}}{10} - \frac{k^{11}}{11}\right)\Bigg|_{0.85}^{0.95} \\ &= 110 \cdot \left(\left(\frac{0.95^{10}}{10} - \frac{0.95^{11}}{11}\right) - \left(\frac{0.85^{10}}{10} - \frac{0.85^{11}}{11}\right)\right) \\ &= 110 \cdot \left(\left(\frac{0.95^{10}}{10} - \frac{0.95^{11}}{11}\right) - \left(\frac{0.85^{10}}{10} - \frac{0.85^{11}}{11}\right)\right) \\ &= 110 \cdot 0.00369017635 \\ &= 0.405919399 \end{aligned}$$

(i) Fill in the blank: you can be 99% sure that $P(H) = k$ is at least ().

$$\begin{aligned}
\int_a^1 p(k)\,\mathrm{d}k &= 0.99 \\
\int_a^1 110 \cdot (k^9 - k^{10})\,\mathrm{d}k &= 0.99 \\
110 \cdot \left(\frac{k^{10}}{10} - \frac{k^{11}}{11}\right)\Bigg|_a^1 &= 0.99 \\
110 \cdot \left(\left(\frac{1^{10}}{10} - \frac{1^{11}}{11}\right) - \left(\frac{a^{10}}{10} - \frac{a^{11}}{11}\right)\right) &= 0.99 \\
110 \cdot \left(0.00909090909 - \left(\frac{a^{10}}{10} - \frac{a^{11}}{11}\right)\right) &= 0.99 \\
1 - 110 \cdot \left(\frac{a^{10}}{10} - \frac{a^{11}}{11}\right) &= 0.99 \\
1 - \left(11a^{10} - 10a^{11}\right) &= 0.99 \\
1 - 11a^{10} + 10a^{11} &= 0.99 \\
10a^{11} - 11a^{10} + 1 &= 0.99 \\
a &= 0.530184
\end{aligned}$$

We can be 99% sure that $P(H) = k$ is at least 0.530184.