Machine Learning Assignment 44

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Problem 19-2

(Part 1)
(1 point per correct answer with supporting work)

a. Write the probability distribution $p_4(n)$ for getting n heads on 4 coin flips, where the coin is a fair coin (i.e. it lands on heads with probability 0.5).

b. Let N be the number of heads in 4 coin flips. Then $N \sim p_4$. Intuitively, what is the expected value of N? Explain the reasoning behind your intuition.

c. Compute the expected value of N, using the definition $E[N] = \sum n \cdot p(n)$. The answer you get should match your answer from (b).

d. Compute the variance of N, using the definition $Var[N] = E[(N - E[N])^2]$. Your answer should come out to 1.

Solution

a.

If we make two "trees" each starting with "H" for heads and "T" for tails, and go through all possible outcomes after flipping the coin 4 times, we should end up with 16 end "branches." Trace those branches and you will get all possible outcomes of four coin flips. Only one of those branches is all heads and another is all tails. So 1/16 is the probability for both 0 heads and 4 heads. We just continue doing this for all of the other number of heads.

So we will end up with:

$p_4(0) = 0.0625$
$p_4(1) = 0.25$
$p_4(2) = 0.375$
$p_4(3) = 0.25$
$p_4(4) = 0.0625$

b.

Intuition tells me that there would be 2 heads. Since we are using a fair coin, the chances of getting a head on a coin flip would be 50 percent (0.50). So I simply think 50 percent of 4 to get that there are two heads in the four coin flips.

c.

E[N] $= \sum n \cdot p(n)$
$= (0 \cdot 0.625) + (1 \cdot 0.25) + (2 \cdot 0.375) + (3 \cdot 0.25) + (4 \cdot 0.625)$
$= 0 + 0.25 + 0.75 + 0.75 + 0.25 = 2$

d.

Var[N] $= E[(N - E[N])^2] = E[(N-2)^2]$
$= (-2^2 \cdot 0.625) + (-1^2 \cdot 0.25) + (0^2 \cdot 0.375) + (1^2 \cdot 0.25) + (2^2 \cdot 0.625)$
$= (4 \cdot 0.625) + (1 \cdot 0.25) + (0 \cdot 0.375) + (1 \cdot 0.25) + (4 \cdot 0.625)$
$= 0.25 + 0.25 + 0 + 0.25 + 0.25 = 1$

(Part 2)
(1 point per correct answer with supporting work)

a. Write the probability distribution $p_{4,k}(n)$ for getting n heads on 4 coin flips, where the coin is a biased coin that lands on heads with probability k. If you substitute $k = 0.5$, you should get the same result that you did in part 1a.

b. Let N be the number of heads in 4 coin flips of a biased coin. Then $N \sim p_{4,k}$. Intuitively, what is the expected value of N? Your answer should be in terms of k. Explain the reasoning behind your intuition. If you substitute $k = 0.5$, you should get the same result that you did in part 1b.

c. Compute the expected value of N, using the definition $E[N] = \sum n \cdot p(n)$. The answer you get should match your answer from (b).

Solution

a.

Let us use the probability "tree" again. We now write out the probabilities on each of the "branches." For example, if the biased coin has a 60 percent chance of landing on heads, each branch going to an "H" would be a 60 percent branch, while every branch going to a "T" would be a 40 percent branch (1-0.6 = 0.4). In the end, when we get all possible results, we will multiply all the probabilities along the branch paths together. For every head along the path, the probability will be multiplied by k, and for every tail, by (1-k). Now that we have the probabilities for all 16 possibilities, we multiply by the number of occurrences of each.

So for 4 flips, we will get:

$p_{4,k}(0) = (1-k)^4$
$p_{4,k}(1) = (k \cdot (1-k)^3) \cdot 4$
$p_{4,k}(2) = (k^2 \cdot (1-k)^2) \cdot 6$
$p_{4,k}(3) = (k^3 \cdot (1-k)) \cdot 4$
$p_{4,k}(4) = k^4$

b.

Last time, my intuition told me that the expected value was 2 because 2 is 50 percent of 4. This time, the 50 percent is replaced by k, so from what I know, my intuition is telling me that the expected value is $4 \cdot k$.

c.

E[N] $= \sum n \cdot p(n)$
$= (0 \cdot (1-k)^4) + (1 \cdot (k \cdot (1-k)^3) \cdot 4) + (2 \cdot (k^2 \cdot (1-k)^2) \cdot 6) + (3 \cdot (k^3 \cdot (1-k)) \cdot 4) + (4 \cdot k^4)$
$= (4k \cdot (1-k)^3) + (12k^2 \cdot (1-k)^2) + (12k^3 \cdot (1-k)) + 4k^4$
$= (4k \cdot (1 - 3l + 3k^2 - k^3)) + (12k^2 \cdot (1 - 2k + k^2)) + (12k^3 - 12k^4) + 4k^4$
$= (4k - 12k^2 + 12k^3 - 4k^4) + (12k^2 - 24k^3 + 12k^4) + (12k^3 - 12k^4) + 4k^4$

Everything besides the 4k at the beginning cancels out, leaving only 4k