Machine Learning Assignment 44

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Problem 20-1

(Part 1)
Consider the exponential distribution defined by
$p_2(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$

a. Using integration, show that this is a valid distribution, i.e. all the probability integrates to 1.

b. Given that $X \sim p_2$*, compute* $P(0 < X \leq 1)$*. You should get a result of* $1 - e^{-2}$*.*

c. Given that $X \sim p_2$*, compute E[X]. You should get a result of* $\frac{1}{2}$*.*

d. Given that $X \sim p_2$*, compute Var[X]. You should get a result of* $1/4$*.*

Solution

a.

$\int_{-\infty}^{\infty} p_2(x)\,dx$

$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} 2e^{-2x}\,dx$

$=\int_{0}^{\infty} 2e^{-2x}\,dx$

$=$ -e$^{-2x}\big|_0^\infty$

$=$ -e$^{-2\infty} + e^0$

$= 1$

b.

$P(0 < X \leq 1)$

$= \int_0^1 p_2(x)\,dx$

$= \int_0^1 2e^{-2x}\,dx$

= -e$^{-2x}\Big|_0^1$

= -e$^{-2} + e^0$

= 1 - e^{-2}

c.

$E[X] = \int_{-\infty}^{\infty} x \cdot p_2(x)\,dx$

$= \int_{-\infty}^{0} x \cdot 0\,dx + \int_0^{\infty} x \cdot 2e^{-2x}\,dx$

$= \int_0^{\infty} 2x \cdot e^{-2x}\,dx$

$= \lim_{a\to\infty}(-xe^{-2x} - \frac{1}{2}(e^{-2x})\Big|_0^a)$

$= \lim_{a\to\infty}(-ae^{-2a} - \frac{1}{2}(e^{-2a}) - (-\frac{1}{2}))$

$= 1/2$

d.

$Var[X] = E[(X - E[X])^2]$

$= E[(X - \frac{1}{2})^2]$

$= \int_{-\infty}^{\infty}(x - \frac{1}{2})^2 \cdot p_2(x)\,dx$

$= \int_{-\infty}^{0}(x - \frac{1}{2})^2 \cdot 0\,dx + \int_0^{\infty}(x - \frac{1}{2})^2 \cdot 2e^{-2x}\,dx$

$= \int_0^{\infty}(x - \frac{1}{2})^2 \cdot 2e^{-2x}\,dx$

(integration by parts and cancelling)

$= \lim_{a\to\infty}(-e^{-2x}(x^2+\frac{1}{4})\big|_0^a)$

$= \lim_{a\to\infty}(-e^{-2a}(a^2+\frac{1}{4})-(-\frac{1}{4}))$

$= 1/4$