# Uniform Distributions

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October 20, 2020

## Problem 21-2

*A uniform distribution on the interval* $[3, 7]$ *is a probability distribution* $p(x)$ *that takes the following form for some constant* $k$*:*

$$p(x) = \begin{cases} k & x \in [3, 7] \\ 0 & x \notin [3, 7] \end{cases}$$

*It is also* $\mathcal{U}[3, 7]$*. So, to say that* $X \sim \mathcal{U}[3, 7]$*, is to say that* $X \sim p$ *for the function* $p$ *shown above.*

***a.*** *Find the value of* $k$ *such that* $p(x)$ *is a valid probability distribution. (Remember that for a function to be a valid probability distribution, it must integrate to 1.)*

***b.*** *Given that* $X \sim \mathcal{U}[3, 7]$*, compute* $E[X]$*. Check: does your result make intuitive sense? If you pick a bunch of numbers from the interval* $[3, 7]$*, and all of those numbers are equally likely choices, then what would you expect to be the average of the numbers you pick?*

***c.*** *Given that* $X \sim \mathcal{U}[3, 7]$*, compute* $Var[X]$*. You should get* $\frac{4}{3}$*.*

**Solution**

a.

$$\begin{aligned}
&\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1 \\
&= \int_{-\infty}^{3} p(x)\,\mathrm{d}x + \int_{3}^{7} p(x)\,\mathrm{d}x + \int_{7}^{\infty} p(x)\,\mathrm{d}x = 1 \\
&= \int_{3}^{7} p(x)\,\mathrm{d}x = 1 \\
&= \int_{3}^{7} k\,\mathrm{d}x = 1
\end{aligned}$$

$= \ kx|_3^7 = 1$

= 7k - 3k = 1

$= \text{k} = \left(\frac{1}{4}\right)$

b.

$$E[X] = \int_{-\infty}^{\infty} x \cdot p(x)\,\mathrm{d}x$$
$$= \int_{-\infty}^{3} x \cdot p(x)\,\mathrm{d}x + \int_{3}^{7} x \cdot p(x)\,\mathrm{d}x + \int_{7}^{\infty} x \cdot p(x)\,\mathrm{d}x$$
$$= \int_{3}^{7} x \cdot p(x)\,\mathrm{d}x$$
$$= \int_{3}^{7} \frac{x}{4}\,\mathrm{d}x$$
$$= \ \frac{1}{8}x^2\Big|_3^7$$
$$= \left(\frac{49}{8}\right) - \left(\frac{9}{8}\right)$$
$$= 5$$

c.

$$Var[X] = E[(X - E[X])^2]$$

= E[(X - 5)$^2$]

$$= \int_{-\infty}^{\infty} (x-5)^2 \cdot p(x)\,\mathrm{d}x$$
$$= \int_{3}^{7} (x-5)^2 \cdot \left(\frac{1}{4}\right)\,\mathrm{d}x$$
$$= \left(\frac{1}{4}\right)\int_{3}^{7} (x^2 - 10x + 25)\,\mathrm{d}x$$
$$= \ \frac{1}{4}(\frac{1}{3}x^3 - 5x^2 + 25x)\Big|_3^7$$
$$= \ \frac{1}{4}\left(\frac{1}{3}(7^3) - 5(7^2) + 25(7) - 39\right)$$

(calculator)

$$= \left(\frac{4}{3}\right)$$