More Distributions

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Problem 23-2

(PART 1)
Consider the general exponential distribution defined by

$$p_\lambda(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a. *Using integration, show that this is a valid distribution, i.e. all the probability integrates to 1.*

b. *Given that $X \sim p_\lambda$, compute $P(0 < X < 1)$.*

c. *Given that $X \sim p_\lambda$, compute $\mathrm{E}[X]$.*

d. *Given that $X \sim p_\lambda$, compute Var$[X]$. Note: Your answers should match those from Assignment 20 when you substitute $\lambda = 2$.*

Solution

a.

$$\begin{aligned}
&\int_{-\infty}^{\infty} p_\lambda(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{0} p_\lambda(x)\,\mathrm{d}x + \int_{0}^{\infty} p_\lambda(x)\,\mathrm{d}x \\
&= \int_{0}^{\infty} p_\lambda(x)\,\mathrm{d}x \\
&= \int_{0}^{\infty} \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= \lim_{a\to\infty} (-e^{-\lambda x}\big|_0^a)
\end{aligned}$$

$$= \lim_{a\to\infty} (-e^{-\lambda a} + 1)$$

$$= 1$$

b.

$$\int_0^1 p_\lambda(x)\,\mathrm{d}x$$

$$= \int_0^1 \lambda e^{-\lambda x}\,\mathrm{d}x$$

$$= -e^{-\lambda x}\Big|_0^1$$

$$= \text{-e}^{-\lambda} + 1$$

c.

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x \cdot p_\lambda(x)\,\mathrm{d}x$$

$$= \int_0^{\infty} x \cdot p_\lambda(x)\,\mathrm{d}x + \int_0^{\infty} x \cdot p_\lambda(x)\,\mathrm{d}x$$

$$= \int_0^{\infty} x \cdot p_\lambda(x)\,\mathrm{d}x$$

$$= \int_0^{\infty} \lambda x e^{-\lambda x}\,\mathrm{d}x$$

...integration by parts...

$$= \lim_{a\to\infty} (-e^{-\lambda x}(x + \tfrac{1}{\lambda})\Big|_0^a)$$

$$= \lim_{a\to\infty} (-e^{-\lambda a}(a + \tfrac{1}{\lambda}) + (\tfrac{1}{\lambda}))$$

$$= \left(\frac{1}{\lambda}\right)$$

d.

$$\mathrm{Var}[X] = \mathrm{E}[(X - \mathrm{E}[X])^2]$$

$$= \text{E[(X - } \frac{1}{\lambda})^2]$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} (x - \frac{1}{\lambda})^2 \cdot p_\lambda(x) \, \mathrm{d}x \\
&= \int_{-\infty}^{0} (x - \frac{1}{\lambda})^2 \cdot p_\lambda(x) \, \mathrm{d}x + \int_{0}^{\infty} (x - \frac{1}{\lambda})^2 \cdot p_\lambda(x) \, \mathrm{d}x \\
&= \int_{0}^{\infty} (x - \frac{1}{\lambda})^2 \cdot p_\lambda(x) \, \mathrm{d}x \\
&= \int_{0}^{\infty} (x - \frac{1}{\lambda})^2 \cdot \lambda e^{-\lambda x} \, \mathrm{d}x
\end{aligned}
$$

...integration by parts and cancelling...

$$
\begin{aligned}
&= \lim_{a \to \infty} \big(-e^{-\lambda x}(x + \tfrac{1}{\lambda^2}) \big|_0^a \big) \\
&= \lim_{a \to \infty} \big(-e^{-\lambda a}(a + \tfrac{1}{\lambda^2}) + (\tfrac{1}{\lambda^2}) \big) \\
&= \left(\frac{1}{\lambda^2} \right)
\end{aligned}
$$

PART 2

Consider the general uniform distribution on the interval $[a, b]$*. It takes the following form for some constant k:* $p(x) = \begin{cases} k & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$

a. *Find the value of* k *such that* $p(x)$ *is a valid probability distribution. Your answer should be in terms of* a *and* b*.*

b. *Given that* $X \sim p$*, compute the cumulative distribution* $P(X \leq x)$*. Your answer should be a piecewise function:* $P(X \leq x) = \begin{cases} \text{---} & \textit{if } x < a \\ \text{---} & \textit{if } a \leq x \leq b \\ \text{---} & \textit{if } b < x \end{cases}$

textbfc.Given that $X \sim p$*, compute* $\mathrm{E}[X]$*.*

d. *Given that* $X \sim p$*, compute* $Var[X]$*.*

Solution

a.

$$\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1$$

$$= \int_{-\infty}^{a} p(x)\,\mathrm{d}x + \int_{a}^{b} p(x)\,\mathrm{d}x + \int_{b}^{\infty} p(x)\,\mathrm{d}x = 1$$

$$= \int_{a}^{b} p(x)\,\mathrm{d}x = 1$$

$$= \int_{a}^{b} k\,\mathrm{d}x = 1$$

$$= kx|_a^b = 1$$

= bk - ak = 1

= k(b - a) = 1

$$= \text{k} = \left(\frac{1}{b-a}\right)$$

b.

$$P(X \leq x) = \begin{cases} 0 & \text{if } x < a \\ \left(\frac{1}{b-a}\right) & \text{if } a \leq x \leq b \\ 0 & \text{if } b < x \end{cases}$$

c.

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x \cdot p(x)\,\mathrm{d}x$$

$$= \int_{-\infty}^{a} x \cdot p(x)\,\mathrm{d}x + \int_{a}^{b} x \cdot p(x)\,\mathrm{d}x + \int_{b}^{\infty} x \cdot p(x)\,\mathrm{d}x$$

$$= \int_{a}^{b} x \cdot p(x)\,\mathrm{d}x$$

$$= \int_{a}^{b} \left(\frac{x}{b-a}\right)\,\mathrm{d}x$$

$$= \frac{1}{2(b-a)} x^2 \Big|_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{b+a}{2}$$

c.

$$Var[X] = \mathrm{E}[(X - \mathrm{E}[X])^2]$$

$$= \mathrm{E}[(\mathrm{X} - \frac{b+a}{2})^2]$$

$$= \int_{-\infty}^{\infty} (x - \frac{b+a}{2})^2 \cdot p(x)\,\mathrm{d}x$$

$$= \int_{a}^{b} (x - \frac{b+a}{2})^2 \cdot \frac{1}{b-a}\,\mathrm{d}x$$

$$= \frac{1}{b-a} \cdot \int_{a}^{b} (x^2 - (b+a)x + \frac{(b+a)^2}{4})\,\mathrm{d}x$$

$$= \frac{1}{b-a} (\frac{1}{3}x^3 - \frac{(b+a)}{2}x^2 + \frac{(b+a)^2}{4}x)\Big|_a^b$$

$$= \frac{1}{b-a} \left(\frac{b^3}{3} - \frac{b^2(b+a)}{2} + \frac{b(b+a)^2}{4} - \frac{a^3}{3} + \frac{a^2(b+a)}{2} - \frac{a(b+a)^2}{4}\right)$$

...expansion and simplification on paper...

$$= \frac{b^3 - a^3}{12(b-a)} - \frac{ab}{4} \text{ (incorrect)}$$

$$= \frac{(b-a)^2}{12}$$