Time Uniform Distributions

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Problem 26-3

(PART 1)
Suppose that you take a bus to work every day. Bus A arrives at 8am but is x minutes late with $x \sim U(0, 20)$. Bus B arrives at 8:10 but with $x \sim U(0, 10)$. The bus ride is 20 minutes and you need to arrive at work by 8:30.
Remember that $U(a, b)$ means the uniform distribution on $[a, b]$. See problem 23-2 if you need a refresher on exponential distributions.
Recall the formulas for the mean and variance of uniform distributions: If $X \sim \mathcal{U}(a, b)$, then $E[X] = \dfrac{a+b}{2}$ and $Var(X) = \dfrac{(b-a)^2}{12}$. You can use these formulas without any further justification.

a. *If you take bus A, what time do you expect to arrive at work? Justify your answer.*

b. *If you take bus B, what time do you expect to arrive at work? Justify your answer.*

c. *If you take bus A, what is the probability that you will arrive on time to work? Justify your answer.*

d. *If you take bus B, what is the probability that you will arrive on time to work? Justify your answer.*

Solution

a.

We expect to arrive at work at 8:30am if we are to take bus A. This is because the expected time that bus A will be late by is 10 minutes ($\frac{20}{2}$ minutes), and taking the bus at 8:10am will get us to work at 8:30am b.

If we take bus B, we should expect to arrive at work at 8:35am. The expected time that bus B will be late by is 5 minutes ($\frac{10}{2}$ minutes), and the bus ride will take 20 minutes.

c.

The latest bus A can arrive without me arriving late is 8:10am. This means that the maximum minutes late the bus can arrive is 10 minutes. The probability of the bus arriving zero to ten minutes late is 50 percent ($0.5 = \displaystyle\int_0^{10} p_\lambda(x)$), which means that the chances of me getting to work on time with bus A is 50 percent.

d.

The latest bus B can arrive without me getting to work late is 8:10am. Since the bus itself arrives at 8:10am, the max amount of time the bus is allowed to be late is zero minutes. And since both bounds of the integral are the same number (when calculating the probability of $P(0 \leq x \leq 0)$), the probability of arriving on time with bus B is zero percent.

(PART 2)
Continuing the scenario above, there is a third option that you can use to get to work: you can jump into a wormhole and (usually) come out almost instantly at the other side. The only issue is that time runs differently inside the wormhole, and while you're probably going to arrive at the other end very quickly, there's a small chance that you could get stuck in there for a really long time.The number of ***seconds*** *it takes you to come out the other end of the wormhole follows an exponential distribution* $Exp(\lambda = 4)$.
Recall the formulas for the mean and variance of exponential distributions: If $X \sim Exp(\lambda)$, *then* $E[X] = \frac{1}{\lambda}$ *and* $Var(X) = \frac{1}{\lambda^2}$. *You can use these formulas without any further justification.*

a. *How long do you expect it to take you to come out of the wormhole? Justify your answer.*

b. *What's the probability of taking longer than a second to come out of the wormhole? Justify your answer.*

c. *Fill in the blank: the probability of coming out of the wormhole within (blank) seconds is 99.999 percent. Justify your answer.*

d. *Your friend says that you shouldn't use the wormhole because there's always a chance that you might get stuck in it for over a day, and if you use the wormhole often, then that'll probably happen sometime within your lifetime. Is this a reasonable fear? Why or why not? Justify your answer by computing the probability that you'll get stuck in the wormhole for over a day if you use the wormhole 10 times each day for 80 years.*
Hint: It's easier to start by computing the probability that you won't get stuck in the wormhole for over a day on any given trip through the wormhole, and then use that to compute the probability that you won't get stuck in the wormhole for over a day if you use the wormhole 10 times each day for 80 years.

Solution

a.

I would expect to emerge from the wormhole in a fourth of a second since the expected value for a exponential distribution is $\frac{1}{\lambda}$, which in this case would be $\frac{1}{4}$ (seconds).

b.

The probability of coming out of the wormhole after 1 second is $\int_1^\infty p_\lambda(x) = \int_1^\infty 4e^{-4x}$. After some work on paper, this simplifies to e^{-4}, which is roughly 1.8 percent chance of emerging after 1 second.

c.

This problem is just what we did in the previous problem backwards. We want to get the time from the chances we are given. We want to find that 0.001% chance exception. All answers are in the form e^{-4n}, so we first do $0.00001 = e^{-4n}$, then after some work on paper, we end up with $n = -ln(0.00001)/4 = 2.88$. So the blank would be filled in by the number 2.88.

d.

One day is 24 hours, which is 1440 minutes, which is 86400 seconds. If we use the wormhole 10 times each day every day, that will mean that we use it 3650 times each year (not counting leap years), which means that we use it 292000 times in 80 years. The chances of NOT getting stuck in the wormhole (per use) is $\int_0^{86400} 4e^{-4x} = 1 - e^{-345600}$. So the chances of NOT getting stuck in the wormhole if we use the wormhole 10 times each day for 80 years (not including leap years) is $(1 - e^{-345600})^{292000}$. Which, according to my TI-30XS calculator, is 1 (not sure how accurate it is). So (according to my calculator), there should be no problem with using the wormhole 10 times a day for 80 years.