More Coin Probabilities

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Problem 27-3

Suppose you toss a coin 10 *times and get the result HHHHT HHHHH. From this result, you estimate that the coin is biased and generally lands on heads* 90% *of the time. But how sure can you be? Let's quantify it.*

a. *Compute the likelihood* $P(HHHHT\ HHHHH \,|\, k)$ *where* $P(H) = k$. *Remember that the likelihood is just the probability of getting the result HHHHT HHHHH under the assumption that* $P(H) = k$. *Your answer should be expressed in terms of* k.

b. *The likelihood* $P(HHHHT\ HHHHH \,|\, k)$ *can almost be interpreted as a probability distribution for* k. *The only problem is that it doesn't integrate to* 1. *Create a probability distribution* $P(k \,|\, HHHHT\ HHHHH)$ *that is proportional to the likelihood* $P(HHHHT\ HHHHH \,|\, k)$.
In other words, find the function $P(k)$ *such that* $P(k) = c \cdot P(HHHHT\ HHHHH \,|\, k)$ *for some constant* c, *and* $\int_0^1 P(k \,|\, HHHHT\ HHHHH) = 1$.
Note: the distribution $P(k \,|\, HHHHT\ HHHHH)$ *is called the "posterior" distribution because it represents the probability distribution of* k *after we have observed the event HHHHT HHHHH.*
The probability distribution of k *before we observed the event is called the "prior" distribution and in this case was given by* $P(k) \sim \mathcal{U}[0,1]$ *since we did not know anything about whether or not the coin is biased (or how biased it is).*

c. *Using the prior distribution* $P(k) \sim \mathcal{U}[0,1]$, *what was the prior probability that the coin was biased towards heads? In other words, what was* $P(k > 0.5)$?

d. *Using the posterior distribution* $P(k \,|\, HHHHT\ HHHHH)$, *what was the posterior probability that the coin was biased towards heads? In other words, what is* $P(k > 0.5 \,|\, HHHHT\ HHHHH)$?

e. *Compare your answers in parts (c) and (d). Did the probability that the* *coin was biased towards heads increase or decrease, after observing the sequence of flips? Why does this make intuitive sense?f. Using the posterior distribution, what is the most probable value of k? In other words, what is value of k at which $P(k \mid HHHHT\ HHHHH)$ reaches a maximum? Show your work using the first or second derivative test.*

g. *Why does your answer to (f) make sense? What's the intuition here?*

h. *What is the probability that the bias k lies within* 0.05 *of your answer to part (g)? In other words, what is the probability that* $0.85 < k < 0.95$?

i. *Fill in the blank: you can be* 99% *sure that* $P(H)$ *is at least* ___.

Solution

a.

$$P(\text{HHHHT HHHHH} \mid k) = k \cdot k \cdot k \cdot k \cdot (1-k) \cdot k \cdot k \cdot k \cdot k \cdot k = k^9 \cdot (1-k)$$

b.

$$\int_0^1 P(\text{HHHHT HHHHH} \mid k) = \int_0^1 k^9 - k^{10} = \frac{k^{10}}{10} - \frac{k^{11}}{11}\Big|_0^1 = \frac{1}{10} - \frac{1}{11} = \frac{1}{110}$$

$P(k) = 110 \cdot P(\text{HHHHT HHHHH} \mid k)$ (posterior distribution)

c.

$$P(k > 0.5) = \int_{0.5}^1 \mathcal{U}[0,1] = 0.50$$

d.

$$P(k > 0.5 \mid \text{HHHHT HHHHH}) = \int_{0.5}^1 P(k) = \int_{0.5}^1 110k^9 - 110k^{10} = 11k^{10} - 10k^{11}\Big|_{0.5}^1 =$$

$1 - 0.0058594 = 0.99414$ (calculator)

e.

The probability of the coin being biased increased. This makes intuitive sense since there are so many more heads than tails, and intuition tells us that the number of tails should be nearly the number of heads.

f.

$P(k) = 110k^9 - 110k^{10}, \quad P'(k) = 990k^8 - 1100k^9$

$990k^8 - 1100k^9 = 0, \quad k = 0.9$ (extremum at k = 0.9)

$P''(k) = 7920k^7 - 9900k^8, \quad P''(0.9) = -473.5 < 0$ ($k = 0.9$ is a maximum)

According to the posterior solution, 0.90 is the most probable value for k

g.

The solution for (f) makes intuitive sense. 90% of the flips are heads, so the most intuitive thing to say is that the chances for getting heads is 90%.

h.

$P(0.85 < k < 0.95 \,|\, \text{HHHHT HHHHH})$

$$= \int_{0.85}^{0.95} P(k) = \int_{0.85}^{0.95} 110k^9 - 110k^{10} = 11k^{10} - 10k^{11}\Big|_{0.85}^{0.95}$$

$= (11 \cdot 0.95^{10} - 10 \cdot 0.95^{11}) - (11 \cdot 0.85^{10} - 10 \cdot 0.85^{11}) = 0.406$ (calculator)

i.

Since the problem uses the phrase "at least," we can set the upper bound of the integral to be 1.

$$0.99 = \int_a^1 110k^9 - 110k^{10} = 11k^{10} - 10k^{11}\Big|_a^1 = 1 - (11a^{10} - 10a^{11})$$

Using the graph of $10a^{11} - 11a^{10} + 0.01 = 0$, we can see that the roots are -0.4788, 0.5302, and 1.0996. And since we are operating only on the bounds $[0, 1]$, the only eligible root is 0.5302. So we can be 99% sure that $P(H)$ is at least 0.5302