Bayesian Inferences

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Problem 35-1

You assume that the numbers come from a discrete uniform distribution $U\{1, 2, \ldots, k\}$ *defined as follows:*

$$p_k(x) = \begin{cases} \frac{1}{k} & x \in \{1, 2, \ldots, k\} \\ 0 & x \notin \{1, 2, \ldots, k\} \end{cases}$$

a. Compute the likelihood $P(\{1, 17, 8, 25, 3\} \mid k)$.

b. Compute the posterior distribution *by normalizing the likelihood. That is to say, find the constant c such that* $\sum_{k=1}^{\infty} c \cdot P(\{1, 17, 8, 25, 3\} \mid k) = 1$. *Then, the posterior distribution will be* $P(k \mid \{1, 17, 8, 25, 3\}) = c \cdot P(\{1, 17, 8, 25, 3\} \mid k)$.

c. *What is the most probable value of* k*? You can tell this just by looking at the distribution* $p(k)$*, but make sure to justify your answer with an explanation.*

d. *The largest number in the dataset is* 25*. What is the probability that* 25 *is actually the upper bound chosen by your friend?*

e. *What is the probability that the upper bound is less than or equal to* 30*?*

f. *Fill in the blank: you can be* 95% *sure that the upper bound is less than* ___.

Solution

a.

$$\begin{aligned} P(\{1,17,8,25,3\} \mid k) &= P(1) \cdot P(17) \cdot P(8) \cdot P(25) \cdot P(3) \\ &= \frac{1}{k} \cdot \frac{1}{k} \cdot \frac{1}{k} \cdot \frac{1}{k} \cdot \frac{1}{k} \\ &= \frac{1}{k^5} \end{aligned}$$

b.

$$\begin{aligned} \sum_{k=1}^{\infty} c \cdot P(\{1,17,8,25,3\} \mid k) &= c \cdot \sum_{k=1}^{\infty} P(\{1,17,8,25,3\} \mid k) = 1 \\ &= c \cdot \sum_{k=25}^{\infty} \frac{1}{k^5} = 1 \\ &= c \cdot 0.0000006929 = 1 \\ &= c = 1443199.7832 \end{aligned}$$

$P(k \mid \{1,17,8,25,3\}) = 1443199.7832 \cdot P(\{1,17,8,25,3\} \mid k)$

c. The most probable value for k is 25 since the probability only goes down when the value for k increases.

d.

$$\begin{aligned} P(25 \mid \{1,17,8,25,3\}) &= 1443199.7832 \cdot P(\{1,17,8,25,3\} \mid 25) \\ &= 1443199.7832 \cdot \frac{1}{25^5} \\ &= 0.147784 \end{aligned}$$

e.

$$\begin{aligned} P(25 \leq k \leq 30 \mid \{1,17,8,25,3\}) &= \sum_{k=25}^{30} 1443199.7832 \cdot P(\{1,17,8,25,3\} \mid k) \\ &= 0.583439 \text{ (calculations done in repl)} \end{aligned}$$

f. We can be 95% sure that the upper bound is less than 52 (calculations done in repl)