Joint Distributions

Justin Hong

December 1, 2020

Problem 35-2

The joint uniform distribution $\mathcal{U}([a, b] \times [c, d])$ *is a distribution such that all points* (x, y) *have equal probability in the region* $[a, b] \times [c, d]$ *and zero probability elsewhere. So, it takes the form*

$$p(x, y) = \begin{cases} k & (x, y) \in [a, b] \times [c, d] \\ 0 & (x, y) \notin [a, b] \times [c, d] \end{cases}$$

for some constant k.

a. *Find the value of* k *such that* $p(x, y)$ *is a valid probability distribution. Your answer should be in terms of* a, b, c, d.

b. *Given that* $(X, Y) \sim p$, *compute* $E[X]$ *and* $E[Y]$. *You should get* $E[X] = \dfrac{a+b}{2}$ *and* $E[Y] = \dfrac{c+d}{2}$

c. *Geometrically,* $[a, b] \times [c, d]$ *represents a rectangle bounded by* $x = a$, $x = b$, $y = c$, *and* $y = d$. *What is the geometric interpretation of the point* $(E[X], E[Y])$ *in this rectangle?*

Solution

a.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,\mathrm{d}x\,\mathrm{d}y &= 1 \\
\int_{c}^{d}\int_{a}^{b} p(x,y)\,\mathrm{d}x\,\mathrm{d}y &= 1 \\
\int_{c}^{d}\int_{a}^{b} k\,\mathrm{d}x\,\mathrm{d}y &= 1 \\
\int_{c}^{d} kb - ka\,\mathrm{d}y &= 1 \\
(kbd - kad) - (kbc - kac) &= 1 \\
kbd - kad - kbc + kac &= 1 \\
k &= \frac{1}{(d-c)(b-a)}
\end{aligned}$$

b.

$$\begin{aligned}
\mathrm{E}[X] &= \int_{c}^{d}\int_{a}^{b} x \cdot p(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{c}^{d}\int_{a}^{b} \frac{x}{(d-c)(b-a)}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{c}^{d} \frac{b^2-a^2}{2(d-c)(b-a)}\,\mathrm{d}y \\
&= (d-c)\left(\frac{b^2-a^2}{2(d-c)(b-a)}\right) \\
&= \frac{(b-a)(b+a)}{2(b-a)} \\
&= \frac{b+a}{2}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}[Y] &= \int_c^d \int_a^b y \cdot p(x,y) \, \mathrm{d}x \, \mathrm{d}y, \\
&= \int_c^d \int_a^b \frac{y}{(d-c)(b-a)} \, \mathrm{d}x \, \mathrm{d}y, \\
&= \int_c^d (b-a) \left(\frac{y}{(d-c)(b-a)} \right) \mathrm{d}y, \\
&= \int_c^d \frac{y}{(d-c)} \, \mathrm{d}y, \\
&= \frac{d^2 - c^2}{2(d-c)} \\
&= \frac{(d-c)(d+c)}{2(d-c)} \\
&= \frac{d+c}{2}
\end{aligned}
$$

c.

Using the idea that $[a,b] \times [c,d]$ represents a rectangle bounded by $x = [a,b]$ and $y = [c,d]$, we can visualize the geometric interpretation of the point $(\mathrm{E}[X], \mathrm{E}[Y])$ being the center of the rectangle.