Assignment 19

Part 1:

A. Write the probability distribution $\mathbf{p_4(n)}$ for getting n heads on 4 coin flips, where the coin is a fair coin (i.e. it lands on heads with probability 0.5)

If $p_4(n) = \frac{4!}{(n!)((4-n)!)} \cdot \frac{1}{2^4}$, then as n goes from 0 to 4...

$$\begin{aligned}
p_4(0) &= \frac{4!}{(0!)(4!)} \cdot \frac{1}{2^4} = 1 \cdot \frac{1}{2^4} = 0.0625 \\
p_4(1) &= \frac{4!}{(1!)(3!)} \cdot \frac{1}{2^4} = 4 \cdot \frac{1}{2^4} = 0.25 \\
p_4(2) &= \frac{4!}{(2!)(2!)} \cdot \frac{1}{2^4} = 6 \cdot \frac{1}{2^4} = 0.375 \\
p_4(3) &= \frac{4!}{(3!)(1!)} \cdot \frac{1}{2^4} = 4 \cdot \frac{1}{2^4} = 0.25 \\
p_4(4) &= \frac{4!}{(4!)(0!)} \cdot \frac{1}{2^4} = 1 \cdot \frac{1}{2^4} = 0.0625
\end{aligned}$$

B. Let N be the number of heads in 4 coin flips. Then $N \sim p_4$. Intuitively, what is the expected value of N? Explain the reasoning behind your intuition.

Intuitively, the expected value of N is 2, as it is automatically assumed that the coin is fair. This means that half of the flips are heads and half are tails. $\frac{1}{2} \cdot 4 = 2$ which is how many heads are assumed

C. Compute the expected value of N, using the definition $\mathbf{E[N] = \Sigma n \cdot p(n)}$.

$$\begin{aligned}
E[N] &= \Sigma n \cdot p(n) \\
&= (0 \cdot p(0)) + (1 \cdot p(1)) + (2 \cdot p(2)) + (3 \cdot p(3)) + (4 \cdot p(4)) \\
&= (1 \cdot 0.25) + (2 \cdot 0.375) + (3 \cdot 0.25) + (4 \cdot 0.0625) \\
&= 2
\end{aligned}$$

D. Compute the variance of N, using the definition $Var[N] = E[(N - E[N])^2]$.

$$
\begin{aligned}
Var[N] &= E[(N - E[N])^2] \\
&= E[(N - 2)^2] \\
&= \Sigma(n - 2)^2 \cdot p(n) \\
&= ((-2)^2 \cdot p(0) + ((-1)^2 \cdot p(1) + ((0)^2 \cdot p(2) + ((1)^2 \cdot p(3) + ((2)^2 \cdot p(4) \\
&= (4 \cdot 0.25) + (1 \cdot 0.375) + (1 \cdot 0.25) + (4 \cdot 0.0625) \\
&= 1.875
\end{aligned}
$$

Part 2:

A. Write the probability distribution $p_{4,k}(n)$ for getting n heads on 4 coin flips, where the coin is a biased coin that lands on heads with probability k.

If $p_{4,k}(n) = \frac{4!}{(n!)((4-n)!)} \cdot (k)^n \cdot (1-k)^{4-n}$, then as n goes from 0 to 4...

$$p_4(0) = \frac{4!}{(0!)(4!)} \cdot (k)^0 \cdot (1-k)^4 = 1 \cdot (1-k)^4$$
$$p_4(1) = \frac{4!}{(1!)(3!)} \cdot (k)^1 \cdot (1-k)^3 = 4 \cdot (k) \cdot (1-k)^3$$
$$p_4(2) = \frac{4!}{(2!)(2!)} \cdot (k)^2 \cdot (1-k)^2 = 6 \cdot (k)^2 \cdot (1-k)^2$$
$$p_4(3) = \frac{4!}{(3!)(1!)} \cdot (k)^3 \cdot (1-k)^1 = 4 \cdot (k)^3 \cdot (1-k)$$
$$p_4(4) = \frac{4!}{(4!)(0!)} \cdot (k)^4 \cdot (1-k)^0 = 1 \cdot (k)^4$$

B. Let N be the number of heads in 4 coin flips of a biased coin. Then N $\sim$ $\mathbf{p}_{4,k}$. Intuitively, what is the expected value of N? Your answer should be in terms of k. Explain the reasoning behind your intuition.

Logically it should be 4k, or something around that. For each flip you have the chance that the coin landed on heads, k. As there are 4 flips, you have 4k

C. Compute the expected value of N, using the definition $\mathbf{E[N] = \Sigma n \cdot p(n)}$.

$$\begin{aligned} E[N] &= \Sigma n \cdot p(n) \\ &= (0 \cdot p(0)) + (1 \cdot p(1)) + (2 \cdot p(2)) + (3 \cdot p(3)) + (4 \cdot p(4)) \\ &= (1 \cdot 4 \cdot (k) \cdot (1-k)^3) + (2 \cdot 6 \cdot (k)^2 \cdot (1-k)^2) + (3 \cdot 4 \cdot (k)^3 \cdot (1-k) + (4 \cdot (k)^4) \\ &= 4(k)(1-k)^3 + 12(k)^2(1-k)^2 + 12(k)^3(1-k) + 4(k)^4 \\ &= 4k \end{aligned}$$