Machine Learning Assignment 23

Part 1:

A: Using integration, show that this is a valid distribution, i.e. all the probability integrates to 1.

$$
\begin{aligned}
\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x &= \int_{0}^{\infty} p(x)\,\mathrm{d}x + \int_{-\infty}^{0} p(x)\,\mathrm{d}x \\
&= \int_{0}^{\infty} \lambda e^{-\lambda x}\,\mathrm{d}x + \int_{-\infty}^{0} 0\,\mathrm{d}x \\
&= \left[-e^{-2\lambda}\right]_0^{\infty} + 0 \\
&= \left(\frac{1}{-e^{2\infty}}\right) - (-e^{0}) \\
&= 0 - (-1) \\
&= 1
\end{aligned}
$$

B: Given that $X \sim p$, compute $P(0 < X < 1)$.

$$
\begin{aligned}
\int_{0}^{1} p(x)\,\mathrm{d}x &= \int_{0}^{1} \lambda e^{-\lambda x}\,\mathrm{d}x \\
&= \left[-e^{-\lambda x}\right]_0^{1} \\
&= -e^{-\lambda} - (-1) \\
&= 1 - e^{-\lambda}
\end{aligned}
$$

C: Given that $\boldsymbol{X} \sim \boldsymbol{p}$, compute E[$\boldsymbol{X}$]

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x \cdot p(x)\,\mathrm{d}x \\
&= \int_{0}^{\infty} x \cdot p(x)\,\mathrm{d}x + \int_{-\infty}^{0} x \cdot p(x)\,\mathrm{d}x \\
&= \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\,\mathrm{d}x + \int_{-\infty}^{0} x \cdot 0\,\mathrm{d}x \\
&= \left[\left(-\frac{\lambda x e^{-\lambda x} + e^{-\lambda x}}{\lambda}\right)\right]_0^{\infty} + 0 \\
&= \left(-\frac{\lambda(\infty)e^{-\lambda\infty} + e^{-\lambda\infty}}{\lambda}\right) - \left(-\frac{\lambda(0)e^{0} + e^{0}}{\lambda}\right) \\
&= 0 + \frac{1}{\lambda} \\
&= \frac{1}{\lambda}
\end{aligned}$$

D: Given that $\boldsymbol{X} \sim \boldsymbol{p}$, compute Var[$\boldsymbol{X}$]

$$\begin{aligned}
Var[X] &= E\left[(X - E[N])^2\right] \\
&= E\left[\left(X - \frac{1}{\lambda}\right)^2\right] \\
&= \int_{-\infty}^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \cdot p(x)\,\mathrm{d}x \\
&= \int_{0}^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \cdot p(x)\,\mathrm{d}x + \int_{-\infty}^{0} \left(x - \frac{1}{\lambda}\right)^2 \cdot p(x)\,\mathrm{d}x \\
&= \int_{0}^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \cdot \lambda e^{-\lambda x}\,\mathrm{d}x + \int_{-\infty}^{0} \left(x - \frac{1}{\lambda}\right)^2 \cdot 0\,\mathrm{d}x \\
&= \int_{0}^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \cdot \lambda e^{-\lambda x}\,\mathrm{d}x + \int_{-\infty}^{0} 0\,\mathrm{d}x \\
&= \left[\left(-\frac{\lambda^2 x^2 e^{-\lambda x} + e^{-\lambda x}}{\lambda^2}\right) + 0\right]_0^{\infty} \\
&= \left(-\frac{\lambda^2(\infty)^2 e^{-\lambda(\infty)} + e^{-\lambda(\infty)}}{\lambda^2}\right) - \left(-\frac{\lambda^2(0)^2 e^{0} + e^{0}}{\lambda^2}\right) \\
&= 0 + \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

Part 2:

A: Find the value of k such that $p(x)$ is a valid probability distribution. Your answer should be in terms of a and b.

$$
\begin{aligned}
\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x &= \int_{-\infty}^{a} p(x)\,\mathrm{d}x + \int_{a}^{b} p(x)\,\mathrm{d}x + \int_{b}^{\infty} p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{a} 0\,\mathrm{d}x + \int_{a}^{b} k\,\mathrm{d}x + \int_{b}^{\infty} 0\,\mathrm{d}x \\
&= [0 + kx + 0]_a^b \\
&= bk - ak \\
&= (b-a)k \\
k &= \frac{1}{(b-a)}
\end{aligned}
$$

B: Given that $X \sim p$, compute the cumulative distribution $P(X \leq x)$. Your answer should be a piecewise function:

$$
P(X \leq x) = \begin{cases} 0 & \text{if } x < a \\ \frac{1}{(b-a)} & \text{if } a \leq x \leq b \\ 0 & \text{if } b < x \end{cases}
$$

C: Given that $X \sim p$, compute $\mathrm{E}[X]$

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x \cdot p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{a} x \cdot p(x)\,\mathrm{d}x + \int_{a}^{b} x \cdot p(x)\,\mathrm{d}x + \int_{b}^{\infty} x \cdot p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{a} 0\,\mathrm{d}x + \int_{a}^{b} \frac{x}{(b-a)}\,\mathrm{d}x + \int_{b}^{\infty} 0\,\mathrm{d}x \\
&= \left[\frac{x^2}{2(b-a)}\right]_a^b \\
&= \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} \\
&= \frac{(b-a)(b+a)}{2(b-a)} \\
&= \frac{(b+a)}{2}
\end{aligned}
$$

D: Given that $\boldsymbol{X} \sim \boldsymbol{p}$, compute Var[$\boldsymbol{X}$]

$$\begin{aligned}
Var[X] &= E\left[(X - E[X])^2\right] \\
&= E\left[\left(x - \frac{(b+a)}{2}\right)^2\right] \\
&= \int_{-\infty}^{\infty} \left(x - \frac{(b+a)}{2}\right)^2 \cdot p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{a} \left(x - \frac{(b+a)}{2}\right)^2 \cdot p(x)\,\mathrm{d}x + \int_{a}^{b} \left(x - \frac{(b+a)}{2}\right)^2 \cdot p(x)\,\mathrm{d}x + \int_{b}^{\infty} \left(x - \frac{(b+a)}{2}\right)^2 \cdot p(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{a} 0\,\mathrm{d}x + \int_{a}^{b} \left(x - \frac{(b+a)}{2}\right)^2 \cdot \left(\frac{1}{(b-a)}\right)\,\mathrm{d}x + \int_{b}^{\infty} 0\,\mathrm{d}x \\
&= \frac{1}{(b-a)} \cdot \left[\left(\left(\frac{2x-b-a}{2}\right)^3 \cdot \frac{1}{3}\right)\right]_a^b \\
&= \frac{1}{3(b-a)} \cdot \left(\left(\frac{b-a}{2}\right)^3 - \left(\frac{a-b}{2}\right)^3\right) \\
&= \frac{1}{3(b-a)} \cdot \left(\frac{-2a^3 + 6a^2b - 6ab^2 + 2b^3}{8}\right) \\
&= \frac{1}{3(b-a)} \cdot \left(\frac{-2(a-b)^3}{8}\right) \\
&= \frac{1}{3(b-a)} \cdot \frac{(b-a)^3}{4} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$