Assignment 23

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1 Part 1

(a) *Using integration, show that this is a valid distribution, i.e. all the probability integrates to 1.*

Answer The probability distribution must integrate to 1 so

$$1 = \int_0^\infty \lambda e^{-x\lambda} \mathrm{d}x + \int_{-\infty}^0 0 \mathrm{d}x$$

$$= -e^{-x\lambda} \Big|_0^\infty + 0$$

$$= -\frac{1}{e^\infty} + e^0$$

$$= 0 + 1$$

(b) *Given that X P_λ, compute $P(0 < X < 1)$.*

Answer Since X is greater than 1 the probability will be = to

$$\int_0^1 \lambda e^{-x\lambda} \mathrm{d}x$$

$$= -e^{-x\lambda} \Big|_0^1$$

$$= -e^{-\lambda} + e^0$$

$$= 1 - e^{-\lambda}$$

(c) *Given that X P_λ, compute E[X]*

Answer

$$E[X] = \int_{-\infty}^\infty x p(x) \mathrm{d}x$$

Since $p(x) = 0$ when $x < 0$ and $p(x) = \lambda e^{-\lambda x}$ when $x \geq 0$

$$E[X] = \int_0^\infty x\lambda e^{-x\lambda} \mathrm{d}x + \int_{-\infty}^0 0 \mathrm{d}x$$

$$= -\frac{(x\lambda+1)e^{-x\lambda}}{\lambda}\Big|_0^\infty + 0$$

$$= -\frac{1+\lambda\infty}{\lambda e^\infty} + \frac{(0+1)e^0}{\lambda}$$

$$= 0 + \frac{1}{\lambda}$$

(d) *Given that X P_λ, compute Var[X].*

Answer since $\text{Var}[N] = E[(NE[N])^2]$ and $E[N] = 1$ $\text{Var}[N] = E[(N - \frac{1}{\lambda})^2]$

$$= \int_{-\infty}^{\infty} (x - \frac{1}{\lambda})^2 p(x)\mathrm{d}x$$

Since $p(x) = 0$ when $x < 0$ and $p(x) = \lambda e^{-\lambda x}$ when $x \geq 0$

$$Var[X] = \int_0^\infty (x - \frac{1}{\lambda})^2 \lambda e^{-x\lambda}\mathrm{d}x + \int_{-\infty}^0 0\mathrm{d}x$$

$$= \lambda(\int_0^\infty x^2 e^{-x\lambda}\mathrm{d}x + \int_0^\infty -x\lambda e^{-x\lambda}\mathrm{d}x + \int_0^\infty \frac{1}{\lambda^2}e^{-x\lambda}\mathrm{d}x)$$

$$= \lambda(-\frac{(e^{-x\lambda}(2+2x\lambda+x^2\lambda^2))}{\lambda^3}\Big|_0^\infty - \frac{1}{\lambda} + \frac{1}{\lambda^3})$$

$$= \lambda(0 + \frac{2}{\lambda^3} - \frac{1}{\lambda} + \frac{1}{\lambda^3})$$

$$= \frac{3}{\lambda^2} - \frac{\lambda}{\lambda^2}$$

$$= \frac{3-\lambda}{\lambda^2}$$

2 Part 2

(a) *Find the value of k such that p(x) is a valid probability distribution. Your answer should be in terms of a and b.*

Answer The probability distribution must integrate to 1 so

$$1 = \int_a^b k\mathrm{d}x$$

$$= xk|_a^b$$

$$= k(b-a) = 1$$

$$k = \frac{1}{b-a}$$

(b) *Given that $X \sim p$, compute the cumulative distribution $P(Xx)$*

Answer

$$P(X \leq x) = \begin{cases} 0 & \text{if } x < a \\ \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{if } b < x \end{cases}$$

(c) *Given that $X \sim p$, compute $E[X]$*

Answer

$$\begin{aligned} E[X] &= \int_a^b x \frac{1}{b-a} \mathrm{d}x \\ &= \frac{x^2}{2b-2a}\bigg|_a^b \\ &= \frac{b^2}{2b-2a} - \frac{a^2}{2b-2a} \\ &= \frac{b^2-a^2}{2(b-a)} \\ &= \frac{(b+a)(b-a)}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

(d) *Given that $X \sim p$, compute $Var[X]$*

Answer

$$\begin{aligned} Var[X] &= \frac{1}{b-a} \int_a^b (x - \frac{b+a}{2})^2 \mathrm{d}x \\ &= \frac{1}{b+a} \int_{\frac{-b-a}{2}+a}^{\frac{-b-a}{2}+b} u^2 \mathrm{d}u \\ &= \frac{1}{b+a} \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} u^2 \mathrm{d}u \\ &= \frac{1}{b-a}(\frac{u^3}{3}\bigg|_{\frac{a-b}{2}}^{\frac{b-a}{2}}) \\ &= \frac{1}{b-a}(\frac{(b-a)^3}{24} - \frac{(a-b)^3}{24}) \\ &= \frac{1}{b-a}(-\frac{(a-b)^3}{12}) \\ &= \frac{(a-b)^2}{12} \end{aligned}$$