27-3

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Problem a

(a) Compute the likelihood $P(\text{HHHHT HHHHH}|k)$ where $P(H) = k$. Remember that the likelihood is just the probability of getting the result HHHHT HHHHH under the assumption that $P(H) = k$. Your answer should be expressed in terms of k.

$$\begin{aligned} P(\text{HHHHT HHHHH}|k) &= P(H) \cdot P(H) \cdot P(H) \cdot P(H) \cdot P(T) \cdot ... \\ &= k \cdot k \cdot k \cdot k \cdot (1-k) \cdot k \cdot k \cdot k \cdot k \cdot k \\ &= k^9(1-k) \end{aligned}$$

Problem b

The likelihood $P(\textbf{HHHHT HHHHH}|k)$ can almost be interpreted as a probability distribution for k. The only problem is that it doesn't integrate to 1. Create a probability distribution $P(k|\textbf{HHHHT HHHHH})$ that is proportional to the likelihood $P(\textbf{HHHHT HHHHH}|k)$. In other words, find the function $P(k)$ such that

$$P(k|\textbf{HHHHT HHHHH}) = c \cdot P(\textbf{HHHHT HHHHH}|k)$$

for some constant c, and $\int_0^1 P(k|\textbf{HHHHT HHHHH}) = 1$.

$$\begin{aligned} \int_0^1 c(k^9 - k^{10})dk &= 1 \\ c\int_0^1 k^9 - k^{10}dk &= 1 \\ c \cdot \frac{1}{110} &= 1 \\ c &= 110 \\ P(k|\text{HHHHT HHHHH}) &= 110(k^9 - k^{10}) \end{aligned}$$

Problem c

Using the prior distribution $P(k) \sim U[0,1]$, what was the prior probability that the coin was biased towards heads? In other words, what was $P(k > 0.5)$?

$$\begin{aligned} P(k > 0.5) &= \int_{0.5}^{1} 1dx \\ &= 1 - 0.5 \\ &= 0.5 \end{aligned}$$

Problem d

Using the posterior distribution $P(k|\textbf{HHHHT HHHHH})$, what was the posterior probability that the coin was biased towards heads? In other words, what is $P(k > 0.5|\textbf{HHHHT HHHHH})$?

$$\begin{aligned} P(k > 0.5|\text{HHHHT HHHHH}) &= \int_{0.5}^{1} 110(k^9 - k^{10})dx \\ &= 110 \int_{0.5}^{1} k^9 - k^{10}dx \\ &= 0.994 \end{aligned}$$

Problem e

Compare your answers in parts (c) and (d). Did the probability that the coin was biased towards heads increase or decrease, after observing the sequence of flips? Why does this make intuitive sense?

The bias increased, which makes sense when looking at the sequence. If the coin was non-biased, there would be more than just 1 tail.

Problem f

Using the posterior distribution, what is the most probable value of k? In other words, what is value of k at which P(k—HHHHT HHHHH) reaches a maximum? Show your work using the first or second derivative test.

$$\begin{aligned} 110(9k^8 - 10k^9) &= 0 \\ 9k^8 - 10k^9 &= 0 \\ k &= 0.9 \end{aligned}$$

Problem g

Why does your answer to (f) make sense? What's the intuition here?

It makes sense because as we saw, the coin was biased with 9 heads to 1 tail, so it would be a bias of 0.9.

Problem h

What is the probability that the bias k lies within 0.05 of your answer to part (g)? In other words, what is the probability that $0.85 < k < 0.95$?

$$
\begin{aligned}
P(0.85 < k < 0.95|\text{HHHHT HHHHH}) &= \int_{0.85}^{0.95} 110(k^9 - k^{10})dk \\
&= 110 \int_{0.85}^{0.95} k^9 - k^{10}dk \\
&= 110 \cdot 0.00369 \\
&= 0.4059
\end{aligned}
$$

Problem i

Fill in the blank: you can be 99% sure that P(H) is at least ___.

$$
\begin{aligned}
0.99 &= \int_x^1 110(k^9 - k^{10})dk \\
0.009 &= \int_x^1 k^9 - k^{10}dk \\
0.009 &= \frac{10k^{11} - 11k^{10} + 1}{110} \\
k &= 0.53
\end{aligned}
$$