35-2

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The joint uniform distribution $U([a,b] \times [c,d])$ is a distribution such that all points (x,y) have equal probability in the region $[a,b] \times [c,d]$ and zero probability elsewhere. So, it takes the form

$$p(x,y) = \begin{cases} k & (x,y) \in [a,b] \times [c,d] \\ 0 & (x,y) \notin [a,b] \times [c,d] \end{cases}$$

for some constant k.

Problem a

Find the value of k such that $p(x,y)$ is a valid probability distribution.

For the distribution to be valid, it has to satisfy

$$\begin{aligned} \int_c^d \int_a^b p(x,y) dx dy &= 1 \\ \int_c^d \int_a^b k \quad dx dy &= 1 \\ k \cdot \int_c^d \int_a^b 1 \quad dx dy &= 1 \\ k \cdot \int_c^d (b-a) dy &= 1 \\ k \cdot (b-a)(d-c) &= 1 \\ k &= \frac{1}{(b-a)(d-c)} \end{aligned}$$

Problem b

Given that $(X, Y) \sim p$, compute $\mathbf{E}[X]$ and $\mathbf{E}[Y]$.

$$\begin{aligned}
E[X] &= \int_c^d \int_a^b x \cdot p(x,y) dx dy \\
&= \int_c^d \int_a^b \frac{x}{(b-a)(d-c)} dx dy \\
&= \frac{1}{(b-a)(d-c)} \cdot \int_c^d (\frac{b^2}{2} - \frac{a^2}{2}) dy \\
&= \frac{1}{(b-a)(d-c)} \cdot (\frac{db^2}{2} - \frac{da^2}{2} - \frac{cb^2}{2} + \frac{ca^2}{2}) \\
&= \frac{db^2 - da^2 - cb^2 + ca^2}{2(b-a)(d-c)} \\
&= \frac{d(b^2 - a^2) - c(b^2 - a^2)}{2(b-a)(d-c)} \\
&= \frac{(d-c)(b+a)(b-a)}{2(b-a)(d-c)} \\
&= \frac{b+a}{2}
\end{aligned}$$

$$\begin{aligned}
E[Y] &= \int_c^d \int_a^b y \cdot p(x,y) dx dy \\
&= \frac{1}{(b-a)(d-c)} \int_c^d y(b-a) dy \\
&= \frac{1}{(d-c)} \int_c^d y \quad dy \\
&= \frac{1}{d-c} \cdot \frac{d^2 - c^2}{2} \\
&= \frac{(d-c)(d+c)}{2(d-c)} \\
&= \frac{d+c}{2}
\end{aligned}$$

Problem c

Geometrically, $[a, b] \times [c, d]$ represents a rectangle bounded by $x = a$, $x = b$, $y = c$, and $y = d$. What is the geometric interpretation of the point $(E[X], E[Y])$ in this rectangle?

Since the expected value is considered as a probability-weighted average, the point $(E[X], E[Y])$ represents the center of the rectangle on the given interval.